Collect. Math. **60**, 1 (2009), 115–122

Characterization of matrix operators on Orlicz spaces

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Received February 6, 2008. Revised September 10, 2008

Abstract

We give equivalent conditions for tensor product of Orlicz sequence spaces, then we obtain a necessary and sufficient condition for a class of matrix operators acting on Orlicz sequence spaces to be continuous and compact.

1. Introduction

Let

$$\Phi(u) = \int_0^{|u|} \phi(t)dt \quad \text{and} \quad \Psi(v) = \int_0^{|v|} \psi(s)ds$$

be a pair of complementary $N-$functions, i.e., $\phi(t)$ is right continuous, $\phi(0) = 0$, $\phi(t) : 0 \nearrow \infty$ as $t : 0 \nearrow \infty$ and ψ is the right inverse of ϕ. The Orlicz sequence space l^Φ is defined to be the set $\{x = \{x_i\} : \rho_\Phi(\lambda x) = \sum_{i=1}^{\infty} \Phi(\lambda x_i) < \infty$ for some $\lambda > 0\}$. The Luxemburg norm and the Amemyia norm are expressed as

$$\|x\|_{(\Phi)} = \inf\left\{c > 0 : \rho_\Phi(\frac{x}{c}) \le 1\right\}$$

and

$$\|x\|_\Phi = \inf_{k>0} \frac{1}{k}\left[1 + \rho_\Phi(kx)\right],$$

respectively. To simplify notations, we write shortly $l^{(\Phi)}$ for $(l^\Phi, \|\cdot\|_{(\Phi)})$ and l^Φ for $(l^\Phi, \|\cdot\|_\Phi)$. In virtue of the basic inequality that: $\|x\|_{(\Phi)} \le \|x\|_\Phi \le 2\|x\|_{(\Phi)}$ for every $x \in l^\Phi$, the embedding results for the Luxemburg norm are suitable for the Amemyia norm.

Project supported by NSFC:10571054, NSFJ:BK2007049, NSFC:10771154.
Keywords: Orlicz space, l^p-space, Matrix operator.
MSC2000: 46E30.

Denote by $\langle x, y\rangle$ the series $\sum_{i=1}^{\infty} x_i y_i$. Observe the following well known facts about functionals on $l^{(\Phi)}$ and l^{Φ} (see [3] or [8, Theorem 1.2.12]):

$$\|x\|_{\Phi} = \sup\left\{|\langle x, y\rangle| : y \in l^{(\Psi)}, \|y\|_{(\Psi)} \leq 1\right\}$$

and

$$\|x\|_{(\Phi)} = \sup\left\{|\langle x, y\rangle| : y \in l^{\Psi}, \|y\|_{\Psi} \leq 1\right\}.$$

They will be used in this context. The basic facts on Orlicz spaces can be found in [1] or [7, 8].

Suppose $A = (a_{ij})$ is an infinite real matrix. It determines a linear operator $\mathbf{A}$ from $l^{(\Phi_1)}$ into l^{Φ_2} according to the following rule:

$$Ax = \left(\sum_{j=1}^{\infty} a_{1j}t_j, \sum_{j=1}^{\infty} a_{2j}t_j, \cdots\right)^T \in l^{\Phi_2}, \text{ for any } x = (t_1, t_2, \cdots)^T \in l^{(\Phi_1)}.$$

We call such bounded matrix operator is a member of $\mathcal{B}(l^{(\Phi_1)}, l^{\Phi_2})$, in which the set of all compact matrix operators is denoted by $\mathcal{B}_c(l^{(\Phi_1)}, l^{\Phi_2})$. As a particular case, for the operator from l^{p_1} to l^{p_2} $(1 < p_1, p_2 < \infty)$, the characterization of $\mathcal{B}(l^{p_1}, l^{p_2})$ (or $\mathcal{B}_c(l^{p_1}, l^{p_2})$) has remained a problem (see [4, Chapter 7.§ 5, Problem 12]) and attracted a lot of researchers (see [2, 5, 6]). [4] characterized $\mathcal{B}(l^{p_1}, l^{p_2})$ in terms of their elements when p_1 or p_2 is 1 or ∞. [2] answered the question when $p_1 = p_2 = 2$. This paper is devoted to operators on Orlicz spaces as a generalization of the case $1 < p_1, p_2 < \infty$, which means that the results of this paper restricted to the class of the matrix operators between two Lebesgue sequence spaces are also new.

Let $\mathcal{N}$ be the set of natural numbers and Φ be an $N-$function. The Lebesgue matrix space $l^{\Phi}(\mathcal{N} \times \mathcal{N})$ is expressed as the set

$$\left\{A = (a_{ij}) : \rho(\lambda A) = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \Phi(\lambda|a_{ij}|) < \infty \text{ for some } \lambda > 0\right\},$$

and the norms on it thereby similarly defined as that of sequence spaces.

If $B = (b_{ij}) \in l^{\Psi}(\mathcal{N} \times \mathcal{N})$ with Ψ being the complementary $N-$function of Φ, we denote

$$\langle A, B\rangle = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} a_{ij}b_{ij}.$$

Clearly, the following Hölder inequality holds:

$$|\langle A, B\rangle| \leq \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} |a_{ij}||b_{ij}| \leq \|A\|_{(\Phi)}\|B\|_{\Psi}.$$

The tensor product $l^{(\Phi_1)} \otimes l^{(\Phi_2)}$ of $l^{(\Phi_1)}$ and $l^{(\Phi_2)}$ is defined (see [7, p. 179]) as the set of the linear span of $\{x \otimes y : x \in l^{(\Phi_1)}, y \in l^{(\Phi_2)}\}$, where

$$x \otimes y := x^T \cdot y = \begin{pmatrix} t_1 \\ t_2 \\ \vdots \end{pmatrix} (s_1, s_2, \cdots) = \begin{pmatrix} t_1s_1 & t_1s_2 & \cdots \\ t_2s_1 & t_2s_2 & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}$$

for $x = (t_1, t_2, \cdots, t_n, \cdots)$ and $y = (s_1, s_2, \cdots, s_n, \cdots)$.

2. Main results

Let us first give a characterization of the Orlicz sequence space $l^{(\Phi)}$. The following theorem is parallel to [7, Theorem 5.4.2] which dealt with the tensor product of Orlicz functions spaces (spaces on diffuse measure) but left the result on sequence spaces unknown.

Theorem 1

Let Φ_1, Φ_2 and Φ be N-functions. Then the following conditions are equivalent:

(i) $\Phi(\alpha uv) \le \Phi_1(u) \cdot \Phi_2(v)$ *for some* $\alpha > 0$ *and* $0 < u, v \le u_0$;

(ii) *There is a constant* $0 < C < \infty$ *such that*

$$\|x \otimes y\|_{(\Phi)} \le C\|x\|_{(\Phi_1)}\|\|y\|_{(\Phi_2)};$$

(iii) $l^{(\Phi_1)} \otimes l^{(\Phi_2)} \subset l^{(\Phi)}(\mathcal{N} \times \mathcal{N})$.

Proof. (i) $\Rightarrow$ (ii): Let $0 \neq x = (t_1, t_2, \cdots) \in l^{(\Phi_1)}$ and $0 \neq y = (s_1, s_2, \cdots) \in l^{(\Phi_2)}$. Then

$$\rho_{\Phi_1}\left(\frac{x}{\|x\|_{(\Phi_1)}}\right) = \sum_{i=1}^{\infty} \Phi_1\left(\frac{t_i}{\|x\|_{(\Phi_1)}}\right) \le 1$$

and

$$\rho_{\Phi_2}\left(\frac{y}{\|y\|_{(\Phi_2)}}\right) = \sum_{j=1}^{\infty} \Phi_2\left(\frac{s_j}{\|y\|_{(\Phi_2)}}\right) \le 1.$$

Therefore, $t_i/\|x\|_{(\Phi_1)} \le \Phi_1^{-1}(1)$ and $s_j/\|y\|_{(\Phi_2)} \le \Phi_2^{-1}(1)$. Letting $c_1 = \max(\Phi_1^{-1}(1)/u_0, 1)$ and $c_2 = \max(\Phi_2^{-1}(1)/u_0, 1)$, we have

$$\begin{aligned}
\rho_\Phi\left(\frac{\alpha(x \otimes y)}{c_1 c_2 \|x\|_{(\Phi_1)}\|y\|_{(\Phi_2)}}\right) &= \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \Phi\left[\alpha\left(\frac{|t_i|}{c_1\|x\|_{(\Phi_1)}}\right)\left(\frac{|s_j|}{c_2\|y\|_{(\Phi_2)}}\right)\right] \\
&\le \sum_{i=1}^{\infty} \Phi_1\left(\frac{|t_i|}{c_1\|x\|_{(\Phi_1)}}\right) \cdot \sum_{j=1}^{\infty} \Phi_2\left(\frac{|s_j|}{c_2\|y\|_{(\Phi_2)}}\right) \\
&\le \sum_{i=1}^{\infty} \Phi_1\left(\frac{|t_i|}{\|x\|_{(\Phi_1)}}\right) \cdot \sum_{j=1}^{\infty} \Phi_2\left(\frac{|s_j|}{\|y\|_{(\Phi_2)}}\right) \le 1\,.
\end{aligned}$$

Therefore,

$$\rho_\Phi\left(\frac{\alpha(x \otimes y)}{c_1 c_2 \|x\|_{(\Phi_1)}\|y\|_{(\Phi_2)}}\right) \le \rho_\Phi\left(\frac{\alpha(x \otimes y)}{c_1 c_2 \|x\|_{(\Phi_1)}\|y\|_{(\Phi_2)}}\right) \le 1\,,$$

whence

$$\|x \otimes y\|_{(\Phi)} \le \frac{c_1 c_2}{\alpha}\|x\|_{(\Phi_1)}\|y\|_{(\Phi_2)}.$$

Consequently, (ii) holds by putting $C = c_1 c_2/\alpha$.

(ii) $\Rightarrow$ (iii): This follows by the fact that $x \otimes y \in l^{(\Phi)}$ iff $\|x \otimes y\|_{(\Phi)} < \infty$.

(iii) $\Rightarrow$ (i): In fact, if (i) is false, then there exist $a_n \searrow 0, b_n \searrow 0$ such that

$$\Phi\left(\frac{a_n b_n}{n4^n}\right) > \Phi_1(a_n)\cdot\Phi_2(b_n), \qquad n \geq 1\,.$$

Using the convexity of Φ we have

$$\Phi_1(a_n)\cdot\Phi_2(b_n) < \frac{1}{4^n}\Phi\left(\frac{a_n b_n}{n}\right), \qquad n \geq 1\,.$$

Since $\Phi_1(a_n)\cdot\Phi_2(b_n) \searrow 0$, we may assume that $\Phi_1(a_n) \leq 1/2^n, \Phi_2(b_n) \leq 1/2^n$ for $n \geq 1$. Then choose integers K_n and J_n such that

$$\frac{1}{2^{n+1}} < K_n\Phi_1(a_n) \leq \frac{1}{2^n}, \qquad \frac{1}{2^{n+1}} < J_n\Phi_2(b_n) \leq \frac{1}{2^n}\,.$$

Let x_0 and y_0 be defined as:

$$x_0 = \Big(\overbrace{a_1, a_1, \cdots, a_1}^{K_1}, \cdots\cdots, \overbrace{a_n, a_n, \cdots, a_n}^{K_n}, \cdots\cdots\Big),$$

$$y_0 = \Big(\overbrace{b_1, b_1, \cdots, b_1}^{J_1}, \cdots\cdots, \overbrace{b_n, b_n, \cdots, b_n}^{J_n}, \cdots\cdots\Big).$$

Then

$$\rho_{\Phi_1}(x_0) = \sum_{n=1}^{\infty} K_n\Phi_1(a_n) \leq \sum_{n=1}^{\infty}\frac{1}{2^n} = 1\,,$$

$$\rho_{\Phi_1}(y_0) = \sum_{n=1}^{\infty} J_n\Phi_2(b_n) \leq \sum_{n=1}^{\infty}\frac{1}{2^n} = 1\,.$$

Therefore, $x_0 \in l^{(\Phi_1)}, y_0 \in l^{(\Phi_2)}$. On the other hand, observe that

$$x_0 \otimes y_0 = \begin{pmatrix} \overbrace{a_1b_1, \cdots, a_1b_1}^{J_1} & \cdots\cdots & \overbrace{a_1b_n, \cdots, a_1b_n}^{J_n} & \cdots\cdots \\ \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots \\ a_1b_1, \cdots, a_1b_1 & \cdots\cdots & a_1b_n, \cdots, a_1b_n & \cdots\cdots \\ a_2b_1, \cdots, a_2b_1 & \cdots\cdots & a_2b_n, \cdots, a_2b_n & \cdots\cdots \\ \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots \\ a_2b_1, \cdots, a_2b_1 & \cdots\cdots & a_2b_n, \cdots, a_2b_n & \cdots\cdots \\ \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots \end{pmatrix} \begin{matrix} \left.\vphantom{\begin{matrix}a\\a\\a\end{matrix}}\right\} K_1 \\ \left.\vphantom{\begin{matrix}a\\a\\a\end{matrix}}\right\} K_2 \\ \vdots \end{matrix}\,.$$

Given $\varepsilon > 0$, choose $n_0 > 1/\varepsilon$, then we have

$$\begin{aligned} \rho_\Phi(\varepsilon(x_0 \otimes y_0)) &= \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} K_iJ_j\Phi(\varepsilon a_ib_j) \\ &\geq \sum_{n=1}^{\infty} K_nJ_n\Phi\Big(\frac{a_nb_n}{n}\Big) \\ &\geq \sum_{n=n_0}^{\infty} K_nJ_n4^n\Phi_1(a_n)\Phi_2(b_n) \\ &= \sum_{n=n_0}^{\infty}\frac{4^n}{4^{n+1}} = \sum_{n=n_0}^{\infty}\frac{1}{4} = \infty\,. \end{aligned}$$

Consequently, $x_0 \otimes y_0 \notin l^{(\Phi)}$ since $\varepsilon > 0$ is arbitrary, which contradicts (iii) and finish the proof. □

Theorem 2

If $(\Phi_1, \Psi_1), (\Phi_2, \Psi_2)$ and (Φ, Ψ) be pairs of complementary $N-$functions. Then the following conditions are equivalent:

(i) $\Phi(\alpha uv) \leq \Phi_1(u) \cdot \Psi_2(v)$ *for some* $\alpha > 0$ *and* $0 < u, v \leq u_0$;
(ii) $\mathbf{A} \in \mathcal{B}(l^{(\Phi_1)}, l^{\Phi_2})$, *for each* $A = (a_{ij}) \in l^{\Psi}(\mathcal{N} \times \mathcal{N})$;
(iii) $\mathbf{A} \in \mathcal{B}_c(l^{(\Phi_1)}, l^{\Phi_2})$, *for each* $A = (a_{ij}) \in l^{\Psi}(\mathcal{N} \times \mathcal{N})$.

Proof. (i) ⇒ (ii): Let (i) be satisfied. In view of Theorem 1, there is $C > 0$ such that

$$\|x \otimes y\|_{(\Phi)} \leq C\|x\|_{(\Phi_1)}\|y\|_{(\Psi_2)} \tag{1}$$

for all $x \in l^{(\Phi_1)}$ and $y \in l^{(\Psi_2)}$. Therefore, $A = (a_{ij}) \in l^{\Psi}(\mathcal{N} \times \mathcal{N})$ implies that $\|A\|_{\Psi} < \infty$ and we deduce from (1) that

$$\begin{aligned}
\|\mathbf{A}\|_{\mathbf{l}^{(\Phi_1)} \to \mathbf{l}^{\Phi_2}} &= \sup\left\{\|\mathbf{Ax}\|_{\Phi_2} : \|\mathbf{x}\|_{(\Phi_1)} \leq \mathbf{1}\right\} \\
&= \sup\left\{|\langle Ax, y\rangle| : \|x\|_{(\Phi_1)} \leq 1, \|y\|_{(\Psi_2)} \leq 1\right\} \\
&= \sup\left\{\left|\langle A^T, x \otimes y\rangle\right| : \|x\|_{(\Phi_1)} \leq 1, \|y\|_{(\Psi_2)} \leq 1\right\} \\
&\leq \sup\left\{\|A^T\|_{\Psi}\|x \otimes y\|_{(\Phi)} : \|x\|_{(\Phi_1)} \leq 1, \|y\|_{(\Psi_2)} \leq 1\right\} \\
&\leq C\|A\|_{\Psi}\,.
\end{aligned}$$

That is, $\mathbf{A}$ is bounded on $l^{(\Phi_1)}$ into l^{Φ_2}.

(ii) ⇒ (iii): For any given $A = (a_{ij}) \in l^{\Psi}(\mathcal{N} \times \mathcal{N})$, put

$$A_n = \begin{pmatrix}
a_{11} & a_{12} & \cdots & a_{1n} & 0 & \cdots \\
a_{21} & a_{22} & \cdots & a_{2n} & 0 & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
a_{n1} & a_{n2} & \cdots & a_{nn} & 0 & \cdots \\
0 & 0 & \cdots & 0 & 0 & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots
\end{pmatrix}.$$

Then $\mathbf{A_n}$ is a finite dimensional operator for each $n \geq 1$. Since

$$\|\mathbf{A} - \mathbf{A_n}\|_{l^{(\Phi_1)} \to l^{\Phi_2}} \leq C\|A - A_n\|_{\Psi} \to 0$$

as $n \to \infty$, we conclude that A is compact and hence (iii) holds.

(iii) ⇒ (i): Suppose that (iii) holds and that (i) is not satisfied. There exist $u_n \searrow 0$ and $v_n \searrow 0$ such that

$$\Phi\left(\frac{u_n v_n}{4^n}\right) > \Phi_1(u_n)\Psi_2(v_n), \qquad n = 1, 2, \cdots.$$

Therefore, we have from a basic property on $N-$function (see [7, Proposition 2.1.1])

$$\frac{u_n v_n}{4^n} > \Phi^{-1}\left(\Phi_1(u_n)\Psi_2(v_n)\right) > \frac{\Phi_1(u_n)\Psi_2(v_n)}{\Psi^{-1}\left(\Phi_1(u_n)\Psi_2(v_n)\right)},$$

or,

$$\Psi\left(\frac{4^n\Phi_1(u_n)\Psi_2(v_n)}{u_nv_n}\right) < \Phi_1(u_n)\Psi_2(v_n)\,. \tag{2}$$

Without loss of generality, we assume that $\Phi_1(u_n) < \frac{1}{2^n}$ and $\Psi_2(v_n) \le \frac{1}{2^n}$ for every $n \ge 1$. Let us choose for each $n \ge 1$ two integers K_n and J_n such that

$$\frac{1}{2^{n+1}} < K_n u_n^{p_1} \le \frac{1}{2^n}, \qquad \frac{1}{2^{n+1}} < J_n v_n^{q_2} \le \frac{1}{2^n}\,. \tag{3}$$

Next we define

$$A_0 = \begin{pmatrix} A_1 & 0 & 0 & \cdots \\ 0 & A_2 & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & A_n & \cdots \\ \cdots & \cdots & \cdots & \ddots \end{pmatrix},$$

where

$$A_n = \frac{4^n\Phi_1(u_n)\Psi_2(v_n)}{u_nv_n}\begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & \cdots & 1 \end{pmatrix}_{J_n\times K_n} \qquad n \ge 1.$$

Then $A_0 \in l^\Psi(\mathcal{N}\times\mathcal{N})$. In fact, by (2) and (3) we have

$$\begin{aligned}\rho_\Psi(A_0) &= \sum_{n=1}^\infty K_nJ_n\Psi\left(\frac{4^n\Phi_1(u_n)\Psi_2(v_n)}{u_nv_n}\right) \\ &< \sum_{n=1}^\infty K_nJ_n\Phi_1(u_n)\Psi_2(v_n) \le \sum_{n=1}^\infty \frac{1}{4^n} \\ &= \frac{1}{3} < \infty\,.\end{aligned}$$

Finally, we will show that $\mathbf{A}_0 \notin \mathcal{B}_c(l^{p_1}, l^{p_2})$. Put

$$x_0 = \Big(\overbrace{u_1, u_1, \cdots, u_1}^{K_1}, \cdots\cdots, \overbrace{u_n, u_n, \cdots, u_n}^{K_n}, \cdots\cdots\Big)^T$$

and

$$y_0 = \Big(\overbrace{v_1, v_1, \cdots, v_1}^{J_1}, \cdots\cdots, \overbrace{v_n, v_n, \cdots, v_n}^{J_n}, \cdots\cdots\Big)^T.$$

Then by (5) we deduce that

$$\rho_{\Phi_1}(x_0) = \sum_{n=1}^\infty K_n\Phi_1(u_n) \le \sum_{n=1}^\infty \frac{1}{2^n} = 1$$

and

$$\rho_{\Psi_2}(y_0) = \sum_{n=1}^\infty J_n\Psi_2(v_n) \le \sum_{n=1}^\infty \frac{1}{2^n} = 1.$$

Note that

$$A_0x_0 = \left(\overbrace{\frac{K_1 4\Phi_1(u_1)\Psi_2(v_1)}{v_1}, \cdots, }^{J_1} \cdots, \overbrace{\frac{K_n 4^n\Phi_1(u_n)\Psi_2(v_n)}{v_n}, \cdots, }^{J_n} \cdots \right)^T,$$

whence we conclude that

$$\begin{aligned} \|\mathbf{A}_0\|_{(\mathbf{\Phi_1})\to\mathbf{\Phi_2}} &= \sup\left\{\|\mathbf{A}_0\mathbf{x}\|_{\mathbf{\Phi_2}} : \|\mathbf{x}\|_{(\mathbf{\Phi_1})} \le \mathbf{1}\right\} \\ &\ge \|A_0x_0\|_{\Phi_2} = \sup\left\{|\langle Ax_0, y\rangle| : \|y\|_{(\Psi_2)} \le 1\right\} \\ &\ge \langle A_0x_0, y_0\rangle = \sum_{n=1}^{\infty} K_n J_n 4^n \Phi_1(u_n)\Psi_2(v_n) \\ &\ge \sum_{n=1}^{\infty} \frac{1}{4} = +\infty,, \end{aligned}$$

which contradicts condition (iii). In such a way the proof is finished. □

We obtain the corresponding result for operator on l^p spaces mentioned by [4] from the above theorem:

Corollary. *If* $p_1, p_2, p \in (1,\infty)$ *and* $1/p_1 + 1/q_1 = 1/p_2 + 1/q_2 = 1/p + 1/q = 1$, *then the following conditions are equivalent:*

(i) $p \ge \max(p_1, q_2)$;
(ii) $\mathbf{A} \in \mathcal{B}(l^{p_1}, l^{p_2})$, *for each* $A = (a_{ij}) \in l^q(\mathcal{N}\times\mathcal{N})$;
(iii) $\mathbf{A} \in \mathcal{B}_c(l^{p_1}, l^{p_2})$, *for each* $A = (a_{ij}) \in l^q(\mathcal{N}\times\mathcal{N})$.

Proof. The sequence space l^p is generated by the $N-$function $\Phi(u) = |u|^p$, so we reduce to check that (i) is equivalent to $(uv)^p \le \alpha u^{p_1} v^{q_2}$ for some constant $\alpha > 0$ and $0 < u, v \le 1$, which is routine. □

Acknowledgement. The author would like to thank Prof. Z.D. Ren for his advice and help.

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