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Boundedness on inhomogeneous Lipschitz spaces of fractional integrals singular integrals and hypersingular integrals associated to non-doubling measures

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To my sons

Abstract

In the context of a finite measure metric space whose measure satisfies a growth condition, we prove "$T1$" type necessary and sufficient conditions for the boundedness of fractional integrals, singular integrals, and hypersingular integrals on inhomogeneous Lipschitz spaces. We also indicate how the results can be extended to the case of infinite measure. Finally we show applications to Real and Complex Analysis.

1. Introduction. Definitions and statement of the theorems

Let $(\mathrm{X}, \mathrm{d}, \mu)$ be a finite measure metric space whose measure μ satisfies a n-dimensional growth condition, that is, (X, d) is a metric space and μ is a finite Borel measure that satisfies the following condition: there is $n > 0$ and a constant $A > 0$ such that $\mu(B_r) \leq Ar^n$, for all balls B_r of radius r and for all $r > 0$. Note that this condition allows the consideration of non-doubling as well as doubling measures.

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Our results will apply to functions defined on the support of μ, of course the support of μ has to be well defined, where $\mathrm{supp}(\mu)$ is the smallest closed set F such that for all Borel sets E, $E \subset F^c, \mu(E) = 0$. For example, if X is separable then the support of μ is well defined. Furthermore to avoid any confusion we will assume that $X = \mathrm{supp}(\mu)$.

The inhomogeneous Lipschitz-Hölder spaces of order $\beta, 0 < \beta \leq 1$, will be denoted Λ_β and consists of all bounded functions f that satisfy

$$\sup_{x \neq y \in X} \frac{|f(x) - f(y)|}{d^\beta(x, y)} < \infty .$$

The space Λ_β is a Banach space with the norm

$$\|f\|_{\Lambda_\beta} = \sup_{x \in X} |f(x)| + \sup_{x \neq y \in X} \frac{|f(x) - f(y)|}{d^\beta(x, y)} .$$

It will be useful to have a notation for each term in the norm, let

$$\sup(f) = \sup_{x \in X} |f(x)| \ , \text{ and } \ |f|_\beta = \sup_{x \neq y \in X} \frac{|f(x) - f(y)|}{d^\beta(x, y)} .$$

The results in this paper have extensions to the case $\mu(X) = \infty$, but the constants depend on the normalization of the integrals at infinity, we will indicate these extensions after the section on proofs. On the other hand, the case of homogeneous Lipschitz spaces was treated in [2, 3], and [4]. The letter C or c will denote a constant not necessarily the same at each ocurrence.

Let $\Omega = X \times X \backslash \Delta$, where $\Delta = \{(x, y) : x = y\}$. A function $L_\alpha(x, y) : \Omega \to \mathbb{C}$, where $\mathbb{C}$ is the set of complex numbers, will be called a standard fractional integral kernel of order α, $0 < \alpha < 1$, when there are constants B_1 and B_2 such that

(L1) $|L_\alpha(x, y)| \leq \dfrac{B_1}{d^{n-\alpha}(x, y)}$.

(L2) $|L_\alpha(x_1, y) - L_\alpha(x_2, y)| \leq B_2 \dfrac{d^\gamma(x_1, x_2)}{d^{n-\alpha+\gamma}(x_1, y)}$, for some $\gamma, \alpha < \gamma \leq 1$, and $2d(x_1, x_2) \leq d(x_1, y)$.

The fractional integral of order α of a function f in Λ_β is defined by:

$$L_\alpha f(x) = \int L_\alpha(x, y) f(y) d\mu(y) .$$

Note that in particular $L_\alpha(x, y) = \frac{1}{d^{n-\alpha}(x,y)}$ is a standard fractional kernel of order α.

Theorem 1

Let $0 < \alpha < \gamma \leq 1$, $0 < \beta < 1$, and $\alpha + \beta \leq 1$ when $1 < n$ or $\alpha + \beta < n$ when $n \leq 1$. The following statements are equivalent:

a) $L_\alpha 1 \in \Lambda_{\alpha+\beta}$.
b) $L_\alpha : \Lambda_\beta \to \Lambda_{\alpha+\beta}$ *is bounded.*

We define now the singular integral kernels that we will consider in Theorem 2 and Theorem 3. A function $K(x,y):\Omega\to\mathbb{C}$ will be called a standard singular integral kernel when there are constants $C_{1,}C_2$ and a number $\gamma, 0<\gamma\le 1$, such that

(S1) $|K(x,y)| \le \dfrac{C_1}{d^n(x,y)}$

(S2) $|K(x_{1,}y) - K(x_2,y)| \le C_2 \dfrac{d^\gamma(x_1,x_2)}{d^{n+\gamma}(x_1,y)}$, for $2d(x_1,x_2) \le d(x_1,y)$.

Let η be a function in $C^1\,[0,\infty)$ such that $0\le\eta\le 1, \eta(s)=0$ for $0\le s\le 1/2$ and $\eta(s)=1$ for $1\le s$. Let $K_\varepsilon(x,y)=\eta(\frac{d(x,y)}{\varepsilon})K(x,y)$, $\varepsilon>0$ where $K(x,y)$ is a standard singular integral kernel. We will denote T_ε the operator $T_\varepsilon f(x)=\int K_\varepsilon(x,y)f(y)d\mu(y)$.

Theorem 2

Let $K(x,y)$ be a standard singular integral kernel. Let $0<\beta<\min(n,\gamma)$. The following two statements are equivalent:

a) $\|T_\varepsilon 1\|_{\Lambda_\beta}\le C$, *for all* $\varepsilon>0$.

b) $T_\varepsilon:\Lambda_\beta\to\Lambda_\beta$ *are bounded and* $\|T_\varepsilon\|_{\Lambda_\beta\to\Lambda_\beta}\le C'$, *for all* $\varepsilon>0$.

One of the novelties in this Theorem is that the cancellation condition (S3) for all x (see below) follows from part a).

In Theorem 3 we will consider Principal Value Singular Integrals. We will denote by Lip_β the space of classes of measurable functions f for which there is a $g\in\Lambda_\beta$ such that $f=g$ except for a set E that depends on f, with $\mu(E)=0$. The norm of f in Lip_β is defined as $\|f\|_{Lip_\beta}=\|f\|_\infty+|f|_\beta$, where

$$|f|_\beta = \sup_{x\ne y\in X}\frac{|g(x)-g(y)|}{d^\beta(x,y)} = \sup_{x\ne y\in X-E}\frac{|f(x)-f(y)|}{d^\beta(x,y)}.$$

We also need to add the following two conditions on the kernel:

(S3) $\left|\displaystyle\int_{r_1<d(x,y)<r_2} K(x,y)d\mu(y)\right| \le C_3$ for all $0<r_1<r_2<\infty, \mu-a.e$ in x.

(S4) $\displaystyle\lim_{\varepsilon\to 0}\int_{\varepsilon<d(x,y)<1} K(x,y)d\mu(y)$ exists $\mu-a.e$ in x.

The principal value singular integral of a function $f\in Lip_\beta$ is defined by

$$Kf(x)=\lim_{\varepsilon\to 0}\int_{\varepsilon<d(x,y)} K(x,y)f(y)d\mu(y)$$

Theorem 3

Let $K(x,y)$ be a standard singular integral kernel that in addition satisfies (S3) and (S4). Let $0<\beta<\min(n,\gamma)$ and $f\in Lip_\beta$. Then $Kf(x)$ is well defined $\mu-a.e.$ and the following two statements are equivalent:

a) $K1\in Lip_\beta$

b) $K: Lip_\beta\to Lip_\beta$ *is bounded.*

A function $D_\alpha(x,y):\Omega\to\mathbb{C}$ will be called a standard hypersingular kernel of order α, $0<\alpha<1$, when there are constants E_1 and E_2 such that:

(D1) $|D_\alpha(x,y)| \le \dfrac{E_1}{d^{n+\alpha}(x,y)}$,

(D2) $|D_\alpha(x_1,y) - D_\alpha(x_2,y)| \le E_2 \dfrac{d^\gamma(x_1,x_2)}{d^{n+\alpha+\gamma}(x_1,y)}$, for some $\gamma, 0 < \gamma \le 1$, and $2d(x_1,x_2) \le d(x_1,y)$.

The hypersingular integral of order α of a function $f \in \Lambda_\beta$, $\alpha < \beta \le 1$, is defined by:

$$D^\alpha f(x) = \int D_\alpha(x,y)\,[f(y) - f(x)]\,d\mu(y)\,.$$

Note that in particular $D_\alpha(x,y) = \frac{1}{d^{n+\alpha}(x,y)}$ is a standard hypersingular kernel of order α when $X = R^n$ and μ is the Lebesgue measure, we have

$$\int \frac{1}{d^{n+\alpha}(x,y)}\,[f(y) - f(x)]\,dy = c_\alpha(\Delta^{\alpha/2} f)(x)$$

for f sufficiently smooth and $0 < \alpha < 2$.

Theorem 4

Let $0 < \alpha < \beta \le 1$ and $\beta - \alpha < n$. Then D^α: $\Lambda_\beta \to \Lambda_{\beta-\alpha}$ is bounded.

Note that $D^\alpha 1 = 0$ by definition. Also, Theorem 4 and its proof are valid without changes in the case $\mu(X) = \infty$.

2. Proofs

We would like to point out that the proofs are based on classical methods, see for example [9], adjusted to the modern "T1" formulation and to the present general context. For carrying out the proofs we need the following known lemma about measures that satisfy the n-dimensional growth condition.

Lemma

Let (X, d, μ) be a measure metric space such that μ satisfies the n-dimensional growth condition, and $r > 0$. Then

1. $\displaystyle\int_{d(x,y)<r} \frac{1}{d^{n-\delta}(x,y)}\,d\mu(y) \le c_1 r^\delta, \;\; 0 < \delta < n\,.$
2. $\displaystyle\int_{r \le d(x,y)} \frac{1}{d^{n+\delta}(x,y)}\,d\mu(y) \le c_2 r^{-\delta}, \quad 0 < \delta$
3. $\displaystyle\int_{r/2 \le d(x,y)<r} \frac{1}{d^{n}(x,y)}\,d\mu(y) \le c_3\,.$

Proof of the Lemma. The three parts are consequences of the growth condition. To prove part 1, we rewrite the integral as a series, bound each term using the growth condition, and we add the resulting series. In detail we have:

$$\begin{aligned}\int_{d(x,x_o)<r} \frac{1}{d^{n-\delta}(x,x_o)}\, d\mu(x) &= \sum_{k=0}^{\infty} \int_{2^{-k-1}r\le d(x,x_o)<2^{-k}r} \frac{1}{d^{n-\delta}(x,x_o)}\, d\mu(x)\\ &\le \sum_{k=0}^{\infty} \frac{\mu(B_{2^{-k}r}(x_o))}{(2^{-k-1}r)^{n-\delta}} \le A\sum_{k=0}^{\infty} \frac{(2^{-k}r)^n}{(2^{-k-1}r)^{n-\delta}}\\ &= Ar^{\delta}\Big(\frac{2^n}{2^{\delta}-1}\Big).\end{aligned}$$

To prove part 2 we perform a similar estimate:

$$\begin{aligned}\int_{d(x,x_o)\ge r} \frac{1}{d^{n+\delta}(x,x_o)}\, d\mu(x) &= \sum_{k=0}^{\infty} \int_{2^{k}r\le d(x,x_o)<2^{k+1}r} \frac{1}{d^{n+\delta}(x,x_o)}\, d\mu(x)\\ &\le \sum_{k=0}^{\infty} \frac{\mu(B_{2^{k+1}r}(x_o))}{(2^{k}r)^{n+\delta}} \le A\sum_{k=0}^{\infty} \frac{(2^{k+1}r)^n}{(2^{k}r)^{n+\delta}}\\ &= Ar^{-\delta}\Big(\frac{2^n 2^{\delta}}{2^{\delta}-1}\Big).\end{aligned}$$

Finally for part 3 we have:

$$\int_{r/2\le d(x,y)<r} \frac{1}{d^{n}(x,y)} d\mu(y) \le \frac{\mu(B_r(x_o))}{(r/2)^n} \le A2^n.$$

Proof of Theorem 1. Observe first that $1 \in \Lambda_\beta$ and therefore condition b) implies condition a). We will prove now that condition a) implies condition b). We can just consider the case $L_\alpha(x,y)=\frac{1}{d^{n-\alpha}(x,y)}$, because the general case is proven in the same way, and we will denote $L_\alpha = I_\alpha$.

Condition (L1) is clearly valid. To show that condition (L2) is verified, we use the Mean Value Theorem. Consider $2d(x_1,x_2) \le d(x_1,y)$, and $0<\theta<1$ we have:

$$\begin{aligned}\Big|\frac{1}{d^{n-\alpha}(x_1,y)} - \frac{1}{d^{n-\alpha}(x_2,y)}\Big| &\le \sup_{\theta}\Big|(-n+\alpha)(\theta d(x_1,y)+(1-\theta)(d(x_2,y))^{-n+\alpha-1}\Big|\\ |d(x_1,y)-d(x_2,y)| &\le B_2\frac{d(x_1,x_2)}{d^{n-\alpha+1}(x_1,y)}.\end{aligned}$$

Now we will estimate $\sup(I_\alpha f)$. Let $x\in X$. We will use the lemma to obtain

$$\begin{aligned}|I_\alpha f(x)| &\le \int_{d(x,y)<1} \frac{|f(y)|}{d^{n-\alpha}(x,y)} d\mu(y) + \int_{1\le d(x,y)} \frac{|f(y)|}{d^{n-\alpha}(x,y)} d\mu(y)\\ &\le \sup(f)\big(c_1+\mu(X)\big),\end{aligned}$$

and therefore $\sup(I_\alpha f) \leq \sup(f)(c_1 + \mu(X))$. We will estimate next $|I_\alpha f|_{(\beta)}$. We write

$$\begin{aligned}
I_\alpha f(x_1) - I_\alpha f(x_2) &= \int_X \frac{f(y)}{d^{n-\alpha}(x_1,y)} d\mu(y) - \int_X \frac{f(y)}{d^{n-\alpha}(x_2,y)} d\mu(y) \\
&= \int_X \frac{f(y)-f(x_1)}{d^{n-\alpha}(x_1,y)} d\mu(y) + f(x_1) \int_X \frac{1}{d^{n-\alpha}(x_1,y)} d\mu(y) \\
&\quad - \int_X \frac{f(y)-f(x_1)}{d^{n-\alpha}(x_2,y)} d\mu(y) - f(x_1) \int_X \frac{1}{d^{n-\alpha}(x_2,y)} d\mu(y) \\
&= \int_X \frac{f(y)-f(x_1)}{d^{n-\alpha}(x_1,y)} d\mu(y) - \int_X \frac{f(y)-f(x_1)}{d^{n-\alpha}(x_2,y)} d\mu(y) \\
&\quad + f(x_1)\left[I_\alpha 1(x_1) - I_\alpha 1(x_2)\right].
\end{aligned}$$

The last term can be bounded using the hypothesis, and we have

$$|f(x_1)\left[I_\alpha 1(x_1) - I_\alpha 1(x_2)\right]| \leq \; c \sup(f) d^{\alpha+\beta}(x_1, x_2).$$

Let now $r = d(x_1, x_2)$ and $B_{2r}(x_1)$ the ball of radius $2r$ and center x_1. We write

$$\begin{aligned}
\left| \int_X \frac{f(y)-f(x_1)}{d^{n-\alpha}(x_1,y)} d\mu(y) - \int_X \frac{f(y)-f(x_1)}{d^{n-\alpha}(x_2,y)} d\mu(y) \right| &\leq \int_{B_{2r}(x_1)} \frac{|f(y)-f(x_1)|}{d^{n-\alpha}(x_1,y)} d\mu(y) \\
&\quad + \int_{B_{2r}(x_1)} \frac{|f(y)-f(x_1)|}{d^{n-\alpha}(x_2,y)} d\mu(y) \\
&\quad + \int_{B^c_{2r}(x_1)} |f(y)-f(x_1)| \left| \frac{1}{d^{n-\alpha}(x_1,y)} - \frac{1}{d^{n-\alpha}(x_2,y)} \right| d\mu(y) \\
&= J_1 + J_2 + J_3.
\end{aligned}$$

For the first term using the lemma we have

$$J_1 \leq |f|_{(\beta)} \int_{B_{2r}(x_1)} \frac{d^\beta(x_1,y)}{d^{n-\alpha}(x_1,y)} d\mu(y) \leq c\,|f|_{(\beta)}\, r^{\alpha+\beta} = c\,|f|_{(\beta)}\, d^{\alpha+\beta}(x_1,x_2).$$

For the second term we write

$$J_2 \leq |f|_{(\beta)} \int_{B_{3r}(x_2)} \frac{(2r)^\beta}{d^{n-\alpha}(x_2,y)} d\mu(y) \leq c\,|f|_{(\beta)}\, d^{\alpha+\beta}(x_1,x_2).$$

For the third term we use (L2) and the lemma to get

$$J_3 \leq |f|_{(\beta)} \int_{B^c_{2r}(x_1)} \frac{B_2}{d^{n-\alpha-\beta}(x_1,y)} d\mu(y) \leq c\,|f|_{(\beta)}\, d^{\alpha+\beta}(x_1,x_2).$$

Collecting the previous estimates, we have

$$\|I_\alpha f\|_{\Lambda_\beta} \leq C\,\|f\|_{\Lambda_\beta}.$$

This concludes the proof of Theorem 1.

Proof of Theorem 2. Observe first that $1 \in \Lambda_\beta$ and therefore condition b) implies condition a).

Before doing the proof of the theorem and for the sake of completeness, we will show that K_ε satisfies conditions (S1) and (S2) with constants independent of ε.

Condition (S1) is true because η is bounded. To show condition (S2), assume that $2d(x_1, x_2) \le d(x_1, y)$ and consider the following two cases:

Case 1: $1 < \frac{d(x_1,y)}{\varepsilon}$ and $1 < \frac{d(x_2,y)}{\varepsilon}$. In this case $K_\varepsilon(x,y) = K(x,y)$, and therefore (S2) is true with the same constant.

Case 2: $1 \ge \frac{d(x_1,y)}{\varepsilon}$ or $1 \ge \frac{d(x_2,y)}{\varepsilon}$. Assume $1 > \frac{d(x_1,y)}{\varepsilon}$.

We write

$$|K_\varepsilon(x_1,y) - K_\varepsilon(x_2,y)| \le \left|\eta\Big(\frac{d(x_1,y)}{\varepsilon}\Big) - \eta\Big(\frac{d(x_2,y)}{\varepsilon}\Big)\right| |K(x_1,y)| + \left|\eta\Big(\frac{d(x_2,y)}{\varepsilon}\Big)\right| |K(x_1,y) - K(x_2,y)| \,.$$

The first term above is less than or equal to

$$\|\eta'\|_\infty \frac{|d(x_1,y) - d(x_2,y)|}{\varepsilon} |K(x_1,y)| \le \|\eta'\|_\infty \frac{d(x_1,x_2)}{\varepsilon} |K(x_1,y)| \le c\Big(\frac{d(x_1,x_2)}{\varepsilon}\Big)^\gamma |K(x_1,y)| \le c\frac{d^\gamma(x_1,x_2)}{d^{n+\gamma}(x_1,y)} \,.$$

On the other hand, the second term is less than or equal to $c\,|K(x_1,y) - K(x_2,y)| \le c\frac{d^\gamma(x_1,x_2)}{d^{n+\gamma}(x_1,y)}$. If $1 \ge \frac{d(x_2,y)}{\varepsilon}$ the proof is similar.

To show that condition a) implies condition b), the first step is to obtain the cancellation condition (S3) of the kernel, for all $x \in X$.

Observe that for $0 < r_1 < r_2 < \infty$, we have

$$\begin{aligned} T_{r_1}1(x) - T_{r_2}1(x) = &\int_{r_1/2<d(x,y)\le r_1} \eta\Big(\frac{d(x,y)}{r_1}\Big) K(x,y)d\mu(y) \\ &+ \int_{r_1<d(x,y)} K(x,y)d\mu(y) \\ &- \int_{r_2/2<d(x,y)<r_2} \eta\Big(\frac{d(x,y)}{r_2}\Big) K(x,y)d\mu(y) \\ &- \int_{r_2\le d(x,y)} K(x,y)d\mu(y)\,. \end{aligned}$$

Since the left hand side is uniformly bounded in r and x, and also the first and third terms are uniformly bounded because of the growth condition (see lemma), it follows that

$$\text{(S3)} \qquad \left|\int_{r_1<d(x,y)<r_2} K(x,y)d\mu(y)\right| \le C, \text{ for all } x\,.$$

Now, we will estimate $\sup |T_\varepsilon f(x)|$. Observe first that

$$\begin{aligned}
T_\varepsilon f(x) &= \int_{d(x,y)\le 1} K_\epsilon(x,y)f(y)d\mu(y) + \int_{d(x,y)>1} K_\epsilon(x,y)f(y)d\mu(y) \\
&= \int_{d(x,y)\le 1} K_\epsilon(x,y)(f(y)-f(x))d\mu(y) \\
&\quad + f(x)\int_{\varepsilon/2<d(x,y)\le\varepsilon} K_\epsilon(x,y)d\mu(y) f(x)\int_{\varepsilon<d(x,y)\le 1} K(x,y)d\mu(y) \\
&\quad + \int_{d(x,y)>1} K_\varepsilon(x,y)f(y)d\mu(y)\,.
\end{aligned}$$

Since $f\in\Lambda_\beta$, by conditions (S1), (S3) and the lemma we can bound the absolute value of each term above by $\|f\|_{\Lambda_\beta}$ and therefore $\sup_{x\in X}|T_\varepsilon f(x)|\le c\,\|f\|_{\Lambda_\beta}$.

Next, we will estimate $\sup_{x\neq y}\frac{|T_\varepsilon f(x)-T_\varepsilon f(y)|}{d^\beta(x,y)}$. We consider the difference $T_\varepsilon f(x_1)-T_\varepsilon f(x_2)$, and the following decomposition:

$$\begin{aligned}
T_\varepsilon f(x_1)-T_\varepsilon f(x_2) &= \int K_\varepsilon(x_1,y)f(y)d\mu(y) - \int K_\varepsilon(x_2,y)f(y)d\mu(y) \\
&= \int K_\varepsilon(x_1,y)\left[f(y)-f(x_1)\right]d\mu(y) + f(x_1)\int K_\varepsilon(x_1,y)d\mu(y) \\
&\quad - \int K_\varepsilon(x_2,y)\left[f(y)-f(x_1)\right]d\mu(y) - f(x_1)\int K_\varepsilon(x_2,y)d\mu(y) \\
&= \int K_\varepsilon(x_1,y)\left[f(y)-f(x_1)\right]d\mu(y) \\
&\quad + \int K_\varepsilon(x_2,y)\left[f(y)-f(x_1)\right]d\mu(y) \\
&\quad + f(x_1)\left[T_\varepsilon 1(x_1)-T_\varepsilon 1(x_2)\right]\,.
\end{aligned}$$

Observe now that the last term can be estimated using the hypothesis and we have

$$\left|f(x_1)\left[T_\varepsilon 1(x_1)-T_\varepsilon 1(x_2)\right]\right| \le c\sup(f)d^\beta(x_1,x_2)\,.$$

To estimate the first two terms, let $r=d(x_1,x_2)$, we rewrite their sum as follows:

$$\begin{aligned}
&\int_{d(x_1,y)<3r} K_\varepsilon(x_1,y)\left[f(y)-f(x_1)\right]d\mu(y) \\
&\quad + \int_{d(x_1,y)<3r} K_\varepsilon(x_2,y)\left[f(y)-f(x_1)\right]d\mu(y) \\
&\quad + \int_{3r<d(x_1,y)} \left[f(y)-f(x_1)\right]\left[K_\varepsilon(x_1,y)-K_\varepsilon(x_2,y)\right]d\mu(y) = H_1+H_2+H_3\,.
\end{aligned}$$

The absolute value of H_3 can be estimated using condition (S2) as follows,

$$|H_3| \le |f|_\beta\, d^\gamma(x_1,x_2)\int_{3r<d(x_1,y)} \frac{d^\beta(x_1,y)}{d^{n+\gamma}(x_1,y)}d\mu(y) \le c\,|f|_\beta\, d^\beta(x_1,x_2)\,.$$

For $|H_1|$we have

$$|H_1| \leq |f|_\beta \int_{d(x_1,y)<3r} \frac{C_1}{d^{n-\beta}(x_1,y)} d\mu(y) \leq c\,|f|_\beta \, d^\beta(x_1,x_2)\,.$$

Finally to estimate H_2 we write

$$\begin{aligned}
&\int_{d(x_1,y)<3r} K_\varepsilon(x_2,y)\,[f(y)-f(x_1)]\,d\mu(y)\\
&= \int_{d(x_1,y)<3r} K_\varepsilon(x_2,y)\,[f(y)-f(x_2)]\,d\mu(y)\\
&\quad + [f(x_2)-f(x_1)] \int_{d(x_1,y)<3r} K_\varepsilon(x_2,y) d\mu(y) = J_1 + J_2\,.
\end{aligned}$$

For the first term we have

$$|J_1| \leq \int_{d(x_2,y)<4r} \frac{c\,\|f\|_{(\beta)}}{d^{n-\beta}(x_2,y)} d\mu(y) \leq c\,|f|_\beta \, d^\beta(x_1,x_2)\,.$$

To estimate J_2, consider first

$$\begin{aligned}
\int_{d(x_1,y)<3r} K_\varepsilon(x_2,y) d\mu(y) &= \int_{d(x_2,y)<2r} K_\varepsilon(x_2,y) d\mu(y)\\
&\quad + \int_{\{y:d(x_1,y)<3r\}\setminus\{y:d(x_2,y)<2r\}} K_\varepsilon(x_2,y) d\mu(y)\,.
\end{aligned}$$

Observe now that condition (S3) implies

$$\left| \int_{d(x_2,y)<2r} K_\varepsilon(x_2,y) d\mu(y) \right| \leq C_3$$

and using part 3 of the lemma we get

$$\left| \int_{\{y:d(x_1,y)<3r\}\setminus\{y:d(x_2,y)<2r\}} K_\varepsilon(x_2,y) d\mu(y) \right| \leq \int_{\{y:2r<d(x_2,y)<4r\}} |K_\varepsilon(x_2,y)|\, d\mu(y) \leq c$$

therefore

$$|J_2| \leq c\,|f|_\beta \, d^\beta(x_1,x_2)$$

collecting the estimates we have:

$$|K_\varepsilon f(x_1) - K_\varepsilon f(x_2)| \leq c\,\|f\|_{\Lambda_\beta} \, d^\beta(x_1,x_2)$$

and finally

$$\|K_\varepsilon f\|_{\Lambda_\beta} \leq c\,\|f\|_{\Lambda_\beta}\,,$$

with c independent of ε.

Proof of Theorem 3. Observe first that $1 \in Lip_\beta$ and therefore condition b) implies condition a). Let $f \in Lip_\beta$, we will show that

$$Kf(x) = \lim_{\varepsilon\to 0} \int_{\varepsilon<d(x,y)} K(x,y) f(y) d\mu(y)$$

exists $\mu - a.e.$ Assume $\varepsilon < 1$, we can write

$$Kf(x) = \lim_{\varepsilon \to 0} \int_{\varepsilon < d(x,y) < 1} K(x,y) \left[f(y) - f(x) \right] d\mu(y)$$
$$+ f(x) \lim_{\varepsilon \to 0} \int_{\varepsilon < d(x,y) < 1} K(x,y) d\mu(y) + \int_{1 \le d(x,y)} K(x,y) f(y) d\mu(y).$$

Since $f \in Lip_\beta$, the first integral converges absolutely, the limit of the second term exists by condition (S4), and the last integral converges absolutely because the integrand is bounded. Furthermore, we have $\|Kf\|_\infty \le c \|f\|_{Lip_\beta}$.

We will estimate now $Kf(x_1) - Kf(x_2)$ for x_1, x_2 two points for which $Kf(x)$ exists. This part of the proof is very similar to the same part in Theorem 2. We write

$$\begin{aligned} Kf(x_1) - Kf(x_2) &= \lim_{\varepsilon \to 0} \int_{\varepsilon < d(x_1,y)} K(x_1,y) f(y) d\mu(y) \\ &\quad - \lim_{\varepsilon \to 0} \int_{\varepsilon < d(x_2,y)} K(x_2,y) f(y) d\mu(y) \\ &= \lim_{\varepsilon \to 0} \int_{\varepsilon < d(x_1,y)} K(x_1,y) \left[f(y) - f(x_1) \right] d\mu(y) \\ &\quad + f(x_1) \lim_{\varepsilon \to 0} \int_{\varepsilon < d(x_1,y)} K(x_1,y) d\mu(y) \\ &\quad - \lim_{\varepsilon \to 0} \int_{\varepsilon < d(x_2,y)} K(x_2,y) \left[f(y) - f(x_1) \right] d\mu(y) \\ &\quad - f(x_1) \lim_{\varepsilon \to 0} \int_{\varepsilon < d(x_2,y)} K(x_2,y) d\mu(y) \\ &= \lim_{\varepsilon \to 0} \int_{\varepsilon < d(x_1,y)} K(x_1,y) \left[f(y) - f(x_1) \right] d\mu(y) \\ &\quad + \lim_{\varepsilon \to 0} \int_{\varepsilon < d(x_2,y)} K(x_2,y) \left[f(y) - f(x_1) \right] d\mu(y) \\ &\quad + f(x_1) \left[K1(x_1) - K1(x_2) \right] . \end{aligned}$$

Observe now that the last term can be estimated using the hypothesis and we have

$$\left| f(x_1) \left[K1(x_1) - K1(x_2) \right] \right| \le c \|f\|_\infty d^\beta(x_1,x_2) .$$

To estimate the first two terms, let $r = d(x_1,x_2)$, and $\varepsilon < r$, we rewrite them as follows:

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\int_{\varepsilon<d(x_1,y)} K(x_1,y)\,[f(y)-f(x_1)]\,d\mu(y)\\
&\quad+\lim_{\varepsilon\to 0}\int_{\varepsilon<d(x_2,y)} K(x_2,y)\,[f(y)-f(x_1)]\,d\mu(y)\\
&=\lim_{\varepsilon\to 0}\int_{\varepsilon<d(x_1,y)<3r} K(x_1,y)\,[f(y)-f(x_1)]\,d\mu(y)\\
&\quad+\lim_{\varepsilon\to 0}\int_{\{y:\varepsilon<d(x_2,y)\}\cap\{y:d(x_1,y)<3r\}} K(x_2,y)\,[f(y)-f(x_1)]\,d\mu(y)\\
&\quad+\int_{3r<d(x_1,y)}[f(y)-f(x_1)]\,[K(x_1,y)-K(x_2,y)]\,d\mu(y)=H_1+H_2+H_3\,.
\end{aligned}$$

The absolute value of H_3 can be estimated as follows,

$$|H_3|\le |f|_\beta\, d^\gamma(x_1,x_2)\int_{3r<d(x_1,y)}\frac{d^\beta(x_1,y)}{d^{n+\gamma}(x_1,y)}d\mu(y)\le c\,|f|_\beta\, d^\beta(x_1,x_2)\,.$$

For $|H_1|$we have

$$|H_1|\le |f|_\beta\int_{d(x_1,y)<3r}\frac{C_1}{d^{n-\beta}(x_1,y)}d\mu(y)\le c\,|f|_\beta\, d^\beta(x_1,x_2)\,.$$

Finally to estimate H_2 we write

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\int_{\{y:\varepsilon<d(x_2,y)\}\cap\{y:d(x_1,y)<3r\}} K(x_2,y)\,[f(y)-f(x_1)]\,d\mu(y)\\
&=\lim_{\varepsilon\to 0}\int_{\{y:\varepsilon<d(x_2,y)\}\cap\{y:d(x_1,y)<3r\}} K(x_2,y)\,[f(y)-f(x_2)]\,d\mu(y)\\
&\quad+[f(x_2)-f(x_1)]\lim_{\varepsilon\to 0}\int_{\{y:\varepsilon<d(x_2,y)\}\cap\{y:d(x_1,y)<3r\}} K(x_2,y)d\mu(y)=J_1+J_2\,.
\end{aligned}$$

For the first term we have

$$|J_1|\le\int_{d(x_2,y)<4r}\frac{c\,\|f\|_{(\beta)}}{d^{n-\beta}(x_2,y)}d\mu(y)\le c\,|f|_\beta\, d^\beta(x_1,x_2)\,.$$

To estimate the second J_2 consider first

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\int_{\{y:\varepsilon<d(x_2,y)\}\cap\{y:d(x_1,y)<3r\}} K(x_2,y)d\mu(y)=\lim_{\varepsilon\to 0}\int_{\varepsilon<d(x_2,y)<2r} K(x_2,y)d\mu(y)\\
&\quad+\int_{\{y:d(x_1,y)<3r\}\setminus\{y:d(x_2,y)<2r\}} K(x_2,y)d\mu(y)\,.
\end{aligned}$$

Observe now that

$$\left|\lim_{\varepsilon\to 0}\int_{\varepsilon<d(x_2,y)<2r} K(x_2,y)d\mu(y)\right|\le C_3$$

and using part 3 of the lemma we get

$$\left|\int_{\{y:d(x_1,y)<3r\}\setminus\{y:d(x_2,y)<2r\}} K(x_2,y)d\mu(y)\right| \leq \int_{\{y:2r<d(x_2,y)<4r\}} |K(x_2,y)|\, d\mu(y) \leq c$$

therefore

$$|J_2| \leq c\,|f|_\beta\, d^\beta(x_1,x_2)$$

collecting the estimates we have:

$$|Kf(x_1) - Kf(x_2)| \leq c\,\|f\|_{Lip_\beta}\, d^\beta(x_1,x_2)$$

and finally

$$\|Kf\|_{Lip_\beta} \leq c\,\|f\|_{Lip_\beta}\,.$$

This concludes the proof of Theorem 3.

Proof of Theorem 4. We will prove the theorem for $D_\alpha(x,y) = \frac{1}{d^{n+\alpha}(x,y)}$, the general case is identical. Note that the proof is also valid for $\mu(X)=\infty$.

We will estimate first $\sup(D_\alpha f)$ for $f \in \Lambda_\beta$. We write

$$\begin{aligned}|D^\alpha f(x)| &\leq \int_{d(x,y)\leq 1} \frac{|f(y)-f(x)|}{d^{n+\alpha}(x,y)}d\mu(y) + \int_{d(x,y)>1} \frac{|f(y)-f(x)|}{d^{n+\alpha}(x,y)}d\mu(y)\\ &\leq |f|_\beta \int_{d(x,y)\leq 1} \frac{1}{d^{n+\alpha-\beta}(x,y)}d\mu(y) + 2\mu(\bar{X})\sup(f)\,.\end{aligned}$$

Since $0<\alpha<\beta\leq 1$, we use part 1 of the lemma to estimate the integral and we obtain that $D^\alpha f(x)$ is well defined everywhere and

$$\sup(D^\alpha f) \leq c\,\|f\|_{\Lambda_\beta}\,.$$

To estimate $|D^\alpha f|_\alpha$, we consider $r = d(x_1,x_2)$ and write

$$\begin{aligned}D^\alpha f(x_1) - D^\alpha f(x_2) &= \int_{d(x_1,y)\leq 2r} \frac{f(y)-f(x_1)}{d^{n+\alpha}(x_1,y)}d\mu(y)\\ &\quad - \int_{d(x_1,y)\leq 2r} \frac{f(y)-f(x_2)}{d^{n+\alpha}(x_2,y)}d\mu(y)\\ &\quad + \int_{d(x_1,y)>2r} [f(y)-f(x_1)]\left[\frac{1}{d^{n+\alpha}(x_1,y)} - \frac{1}{d^{n+\alpha}(x_2,y)}\right]d\mu(y)\\ &\quad - \int_{d(x_1,y)>2r} \frac{f(x_1)-f(x_2)}{d^{n+\alpha}(x_2,y)}d\mu(y)\,.\end{aligned}$$

Using part 1 of the lemma and the fact that f is in Λ_β we can obtain that each of the first two terms converges absolutely and is bounded by $c\,|f|_\beta\, d^{\beta-\alpha}(x_1,x_2)$. Using part 2 of the lemma we can also obtain that the fourth term converges absolutely and is bounded by $c\,|f|_\beta\, d^{\beta-\alpha}(x_1,x_2)$.

To estimate the third term observe first that for $2d(x_1,x_2)\leq d(x_1,y)$,

$$\left|\frac{1}{d^{n+\alpha}(x_1,y)}-\frac{1}{d^{n+\alpha}(x_2,y)}\right| \le \sup_\theta \left|(-n-\alpha)(\theta d(x_1,y)+(1-\theta)(d(x_2,y))^{-n-\alpha-1}\right|$$
$$\times\left|d(x_1,y)-d(x_2,y)\right| \le c\frac{d(x_1,x_2)}{d^{n+\alpha+1}(x_1,y)}\,.$$

Therefore using this estimate, the fact that $f\epsilon\Lambda_\beta$ and the part 2 of lemma we obtain that the third term converges absolutely and is less than or equal to $c\,|f|_\beta\, d^{\beta-\alpha}(x_1,x_2)$ and consequently $|D^\alpha f|_{(\beta-\alpha)} \le c\,|f|_\beta$.

Finally combining the two estimates we get $\|D^\alpha f\|_{\Lambda_{\beta-\alpha}} \le c\,\|f\|_{\Lambda_\beta}$.

To extend Theorem 1 and Theorem 3 to the case $\mu(X)=\infty$, the fractional integrals and singular integrals have to be redefined so they converge for $d(x,y)>1$. The operator's norm in each result will depend on the normalization. We will denote with ' the normalizations. Let $x_o \in X$ be a fixed point for which (S4) is valid and define:

$$L_\alpha^{'} f(x) = \int \left[L_\alpha(x,y)-L_\alpha(x_o,y)\right] f(y)d\mu(y)$$
$$K'f(x)=\lim_{\varepsilon\to 0}\int_{\epsilon<d(x,y)}\left[K(x,y)-K(x_o,y)\right] f(y)d\mu(y)\,.$$

Applications

In this section we will illustrate some applications of the theorems. I am indebted to Joaquim Bruna for pointing out to me the Theorem of Mark Krein and to Joan Verdera for several generous discussions on applications 1 and 2.

1. The purpose of this application is to obtain boundedness in L^2 of some singular integrals in the context of non-doubling measure metric spaces of finite measure. Following [7], a singular integral associated to μ is said to be bounded in L^2 when there is a constant C such that $\|K_\varepsilon f\|_{L^2} \le C\,\|f\|_{L^2}$, for all $\varepsilon>0$, where $K_\varepsilon f(x)=\int_{d(x,y)>\varepsilon} K(x,y)f(y)d\mu(y)$. We will use Theorem 2 and the following Theorem of Mark Krein (see [1] for its proof and application to the classical case, and [8] for the case of spaces of homogeneous type):

M. Krein's Theorem

Let H be a real or complex Hilbert space with inner product $(.,.)$ and norm $\|.\|_H$. Let $D\subset H$ be a Banach space dense in H and such that $\|x\|_H \le C\,\|x\|_D$ for $x\in D$. Let A and B be two linear operator such that $\|Ax\|_D \le C_A\,\|x\|_D$, $\|Bx\|_D \le C_B\,\|x\|_D$, $x\in D$ and $(Ax,y)=(x,By)$ for all $x,y\in D$. Then $\|Ax\|_H \le (C_AC_B)^{1/2}\,\|x\|_H$, $\|Bx\|_H \le (C_AC_B)^{1/2}\,\|x\|_H$, $x\in D$, and both extend to bounded operator on H.

In our application, we will consider $H=L^2$ and $D=\Lambda_\beta$. Since X has finite measure we clearly have $\|f\|_{L^2} \le \mu(X)^{1/2}\,\|f\|_{\Lambda_\beta}$, but we need the extra assumption Λ_β dense in L^2. Let now $K(x,y)$ be a standard singular integral kernel and $K^*(x,y)=\overline{K(y,x)}$. Assume that $K^*(x,y)$ also satisfies (S2). Let $A=T_\varepsilon$ and $B=T_\varepsilon^*$ the corresponding smooth truncations. If $\|T_\varepsilon 1\|_{\Lambda_\beta} \le C'$ and $\|T_\varepsilon^* 1\|_{\Lambda_\beta} \le C''$ for all

$\varepsilon > 0$, then by Theorem 2 both T_ε and T_ε^* are uniformly bounded on Λ_β, by Krein's Theorem there is C such that $\|T_\varepsilon f\|_{L^2} \le C\,\|f\|_{L^2}$, $f \in \Lambda_\beta$, for all $\varepsilon > 0$. Consequently $\|K_\varepsilon f\|_{L^2} \le C\,\|f\|_{L^2}$, $f \in \Lambda_\beta$ for all $\varepsilon > 0$, and it extends to a bounded operator in L^2,same conclusion for K^*. In addition, Nazarov, Treil, and Volberg have extended the classical result of Calderon-Zygmund on the boundedness in L^p, $1 < p < \infty$, of singular integrals bounded in L^2, to non-doubling separable measure metric spaces, see [6].

2. The second application has appeared in [5] and it is an example for Theorem 3. In this paper the authors need to study the boundedness properties of the Restricted Beurling Transform, $B_\Omega f = B(f\chi_\Omega)$, on $Lip_\varepsilon(\Omega)$ where Ω is a bounded domain in R^n with boundary of class $C^{1+\varepsilon}$, $0 < \varepsilon < 1$. Mateu, Orobitg and Verdera prove the following more general result: "Let Ω be a bounded domain with boundary of class $C^{1+\varepsilon}$,$0 < \varepsilon < 1$, and let T be an even smooth homogeneous Calderon-Zygmund operator. Then T_Ω maps $Lip_\varepsilon(\Omega)$ into $Lip_\varepsilon(\Omega)$ and also $Lip_\varepsilon(\Omega)$ into $Lip_\varepsilon(\Omega^c)$". Their proof, which is non-trivial, consists in showing that condition (S3) and part a) of Theorem 3 above are met. Condition (S4) is known to be true in this case.

3. The third application is related to M. Riesz Fractional Calculus associated to non-doubling measures. Applying Theorem 1 and Theorem 4 we can obtain that the composition of a Riesz fractional integral $I_\alpha f(x) = \int \frac{1}{d^{n-\alpha}(x,y)} f(y)d\mu(y)$ and a fractional derivative $D^\alpha f(x) = \int \frac{[f(y)-f(x)]}{d^{n+\alpha}(x,y)} d\mu(y)$ of the same order $D^\alpha I_\alpha$, as well as its transpose $I_\alpha D^\alpha$, are bounded on Λ_β, when $I_\alpha 1 \in \Lambda_{\alpha+\beta}, \alpha + \beta < 1$. In addition, it was shown in [4] that these compositions are singular integral operators associated to μ.

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