Regularity bounds by minimal generators and Hilbert function

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ABSTRACT

Let ρ_C be the regularity of the Hilbert function of a projective curve C in $\mathbf{P}_K^n$ over an algebraically closed field K and $\beta_1, \dots, \beta_{n-1}$ be degrees for which there exists a complete intersection of type $(\beta_1, \dots, \beta_{n-1})$ containing properly C . Then the Castelnuovo-Mumford regularity of C is bounded above by $\max\{\rho_C + 1, \beta_1 + \dots + \beta_{n-1} - (n - 1)\}$. We investigate the sharpness of the above bound, which is achieved by curves algebraically linked to ones having degenerate general hyperplane section.

1. Introduction

Estimates for the Castelnuovo-Mumford regularity $\text{reg}(X)$ of a projective scheme X in $\mathbf{P}^n$ give a lot of information on X and it is interesting to predict such estimates from the Hilbert function H_X of X , involving in particular the regularity ρ_X of H_X . In this context, we focus attention mainly on the case where X is a curve.

To address the question, let $\mathbf{P}_K^n$ be a projective space of dimension n over an algebraically closed field K and C be a curve in $\mathbf{P}_K^n$, i.e. a projective scheme of pure dimension 1. On the base of results of [8] applied to the general hyperplane section Z of C , we observe that, if $\beta_1, \dots, \beta_{n-1}$ are degrees for which there exists a complete

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intersection (c.i. for short) of type $(\beta_1, \dots, \beta_{n-1})$ containing properly the curve C , then (Proposition 3.4)

$$\operatorname{reg}(C) \leq \max \{ \rho_C + 1, \beta_1 + \dots + \beta_{n-1} - (n-1) \}. \quad (1.1)$$

Note that the integers $\beta_1, \dots, \beta_{n-1}$ can be chosen among the degrees of minimal generators of the defining ideal $I(C)$ of C . In the literature bounds in terms of the degrees of defining equations of projective schemes are given for smooth [3] and locally complete intersection [6] schemes, and for arbitrary schemes [5]. In [15, 17] bounds for space curves are given by degrees of generators of the defining ideal together with other invariants.

The paper is organized as it follows. In Section 2 we fix notation and recover a bound for the Castelnuovo-Mumford regularity of projective schemes by the study of hyperplane sections (Lemma 2.6). In Section 3, we show how formula (1.1) raises from results of [8] and the equality holds for curves which are algebraically linked to ones having degenerate general hyperplane section (Proposition 3.4). From ideas of [19], a bound for the regularity of any equidimensional locally Cohen-Macaulay (for short ICM) scheme X follows (Remark 3.5). Then, we look after space curves (Corollaries 3.6 and 3.7) and obtain that, in characteristic 0, formula (1.1) is sharp for a ICM curve $C \subset \mathbf{P}_K^3$ with $\operatorname{reg}(C) > \rho_C + 1$ if, and only if, C is algebraically linked to a plane curve (Theorem 3.9). In Proposition 3.11 a bound for integral space curves deriving easily by the shape of Borel ideals is stated. In Section 4 we provide examples of curves for which the given bounds are sharp, computing them either by hand with basic double linkage techniques, or, by **Points** [21] and **CoCoA** [7].

2. General setting

Let K be an algebraically closed field, $S = K[x_0, \dots, x_n]$ the polynomial ring over K in $n+1$ variables, $\mathbf{P}_K^n = \operatorname{Proj} S$ the projective space of dimension n over K .

If $I \subset S$ is a homogeneous ideal, I_t and $I_{\leq t}$ denote the subsets of I consisting of the homogeneous polynomials of degrees t and $\leq t$, respectively. Moreover, α_I is the *initial degree* of I . Finally, $h \in S_1$ is a general linear form which is not a zero-divisor on S/I^{sat} .

We will use freely the common notation of sheaf cohomology in $\mathbf{P}_K^n$, referring to [9, 11, 16] for notation and basic results.

DEFINITION 2.1 The *saturation* of a homogeneous ideal I is $I^{\operatorname{sat}} = \{f \in S \mid \forall j = 0, \dots, n, \exists r \in \mathbf{N} : x_j^r f \in I\}$. The ideal I is *saturated* if $I^{\operatorname{sat}} = I$ and is *m-saturated* if $(I^{\operatorname{sat}})_t = I_t$ for each $t \geq m$. The *satiety* of I is $\operatorname{sat}(I) = \min \{t \mid I \text{ is } t\text{-saturated}\}$.

DEFINITION 2.2 (a) A finitely generated graded S -module M is *m-regular* if the i -th syzygy module of M is generated in degree $\leq m+i$, for all $i \geq 0$. The *regularity* $\operatorname{reg}(M)$ of M is the smallest integer m for which M is m -regular.

(b) A coherent sheaf $\mathcal{F}$ on $\mathbf{P}^n$ is *m-regular* if $H^i(\mathcal{F}(m-i)) = 0$ for all $i > 0$. The *Castelnuovo-Mumford regularity* (or *regularity*) $\operatorname{reg}(\mathcal{F})$ of $\mathcal{F}$ is the smallest integer m for which $\mathcal{F}$ is m -regular.

Remark 2.3 Let I be a homogeneous ideal of S . If ω_I is the maximal degree of minimal generators of I , then $\omega_I \leq \text{reg}(I)$. Moreover, since a homogeneous ideal I is m -regular if and only if I is m -saturated and its sheafification $\mathcal{I}$ is m -regular (see e.g. [9, Proposition 2.6]), the regularity of a saturated homogeneous ideal I equals the regularity of its sheafification and, if $X \subset \mathbf{P}_K^n$ is a closed subscheme, its *regularity* $\text{reg}(X)$ is defined as the regularity $\text{reg}(I)$ of its defining ideal I .

For a finitely generated graded S -module M we let $H_M(t) := \dim_K M_t$ be the *Hilbert function* of M , $\Delta H_M(t) := H_M(t) - H_M(t-1)$, for $t \geq 1$, and $\Delta H_M(0) := 1$. We recall that, for $t \gg 0$, $H_M(t) = P_M(t)$ where $P_M(z) \in K[z]$ is the *Hilbert polynomial* of M . The *regularity* of H_M is $\rho_M := \min \{ \bar{t} / H_M(t) = P_M(t), \forall t \geq \bar{t} \}$.

If I is the defining ideal of a closed subscheme $X \subset \mathbf{P}_K^n$ of dimension k , instead of $H_{S/I}$, $P_{S/I}$, $\rho_{S/I}$ we can write H_X , P_X , ρ_X . Recall that the Hilbert polynomial of X is also $P_X(t) = \sum_{i=0}^k (-1)^i h^i(\mathcal{O}_X(t))$ (see [11, Exercise III.5.2]).

From now on, let $X \subset \mathbf{P}_K^n$ be a scheme of dimension $k > 0$, I its defining ideal and $J = (I, h)$. Moreover, let $Z \subset \mathbf{P}_K^{n-1}$ be the scheme of dimension $k-1$ defined by the saturated ideal $J^{\text{sat}}/(h) = (I, h)^{\text{sat}}/(h)$, i.e. the general hyperplane section of X .

Remark 2.4 From the short exact sequences $0 \rightarrow \mathcal{I}_X(t) \rightarrow \mathcal{O}_{\mathbf{P}^n}(t) \rightarrow \mathcal{O}_X(t) \rightarrow 0$ and $0 \rightarrow \mathcal{I}_X(t-1) \rightarrow \mathcal{I}_X(t) \rightarrow \mathcal{I}_Z(t) \rightarrow 0$ we get, respectively, the long exact sequences

$$\begin{aligned} 0 \rightarrow H^0(\mathcal{I}_X(t)) \rightarrow H^0(\mathcal{O}_{\mathbf{P}^n}(t)) \rightarrow H^0(\mathcal{O}_X(t)) \rightarrow H^1(\mathcal{I}_X(t)) \\ \rightarrow H^1(\mathcal{O}_{\mathbf{P}^n}(t)) \rightarrow H^1(\mathcal{O}_X(t)) \rightarrow H^2(\mathcal{I}_X(t)) \rightarrow \dots \rightarrow H^n(\mathcal{O}_{\mathbf{P}^n}(t)) \rightarrow 0 \end{aligned} \quad (2.1)$$

and

$$\begin{aligned} 0 \rightarrow H^0(\mathcal{I}_X(t-1)) \rightarrow H^0(\mathcal{I}_X(t)) \rightarrow H^0(\mathcal{I}_Z(t)) \rightarrow X \\ \rightarrow H^1(\mathcal{I}_X(t-1)) \rightarrow H^1(\mathcal{I}_X(t)) \rightarrow H^1(\mathcal{I}_Z(t)) \rightarrow \dots \end{aligned} \quad (2.2)$$

From the sequence (2.1) it follows that:

- (i) $H_X(t) = h^0(\mathcal{O}_{\mathbf{P}^n}(t)) - h^0(\mathcal{I}_X(t)) = h^0(\mathcal{O}_X(t)) - h^1(\mathcal{I}_X(t))$;
 - (ii) $h^{i+1}(\mathcal{I}_X(t)) = h^i(\mathcal{O}_X(t))$ for $1 \leq i \leq k$ and for $t \geq -n$, $h^i(\mathcal{I}_X(t)) = 0$ for every $i \geq k+2$;
 - (iii) $\text{reg}(X) \geq \rho_X + 1$;
 - (iv) when $k = 1$ we have $h^2(\mathcal{I}_X(t)) = h^1(\mathcal{I}_X(t))$ for every $t \geq \rho_X$;
- and, also from the sequence (2.2), it follows that
- (v) $h^i(\mathcal{I}_X(t)) = 0$, for every $t \geq \text{reg}(Z) - 1$ and $i \geq 2$ (see, for example, [4, Proposition 5.1]);
 - (vi) $h^1(\mathcal{I}_X(t))$ is strictly decreasing for $t \geq \text{reg}(Z)$.

Hence, when $\rho_X + 1 > \text{reg}(Z)$, $\text{reg}(X)$ depends only on the vanishing of $h^1(\mathcal{I}_X(t))$. In particular, if $h^1(\mathcal{I}_X(\text{reg}(Z) - 1)) = 0$, then $\text{reg}(X) = \text{reg}(Z)$; otherwise, $\text{reg}(X) = \rho_X + 1 > \text{reg}(Z)$.

Remark 2.5 (a) From [2, Lemma 1.10] it follows immediately that $\text{reg}(I) = \max \{ \text{sat}(I), \text{reg}((I, h)) \}$. Thus, if X is an arithmetically Cohen-Macaulay (for short aCM) scheme of dimension k , we have $\text{reg}(X) = \rho_X + k + 1$.

(b) By the short exact sequence

$$0 \longrightarrow (S/I)_{t-1} \xrightarrow{\cdot h} (S/I)_t \longrightarrow (S/J)_t \longrightarrow 0$$

it follows that $H_{S/J}(t) = \Delta H_{S/I}(t)$ and then $\rho_{S/J} = \rho_X + 1$. Note that $H_{S/J}(t) = H_Z(t)$, for every $t \geq \max\{\rho_Z, \rho_{S/J}\}$, hence $\text{sat}(J) \leq \max\{\rho_Z, \rho_{S/J}\}$.

The following result, obtained here by an easy computation involving only Hilbert functions and saturated ideals, is also consequence of cohomological properties.

Lemma 2.6

With the above notation, $\text{reg}(X) = \max\{\rho_X + 1, \text{reg}(Z)\}$. Namely,

- (1) $\rho_X + 1 \geq \text{reg}(Z) \Rightarrow \text{sat}(J) = \text{reg}(X) = \rho_X + 1$,
- (2) $\rho_X + 1 = \text{reg}(Z) - 1 \Rightarrow \text{sat}(J) \leq \text{reg}(Z) - 1$ and $\text{reg}(X) = \text{reg}(Z)$,
- (3) $\rho_X + 1 < \text{reg}(Z) - 1 \Rightarrow \text{sat}(J) = \text{reg}(Z) - 1$ and $\text{reg}(X) = \text{reg}(Z)$.

Proof. By Remarks 2.3 and 2.5(a) it follows that $\text{reg}(X) = \text{reg}(I) = \text{reg}(J) = \max\{\text{sat}(J), \text{reg}(J^{\text{sat}})\}$. Hence, $\text{reg}(X) = \max\{\text{sat}(J), \text{reg}(Z)\}$.

If $\rho_X + 1 \geq \text{reg}(Z)$ then $\rho_{S/J} = \rho_X + 1 \geq \rho_Z + 1$, by Remarks 2.4(iii) and 2.5(b). Hence, for every $t \geq \rho_{S/J} \geq \rho_Z + 1$ we have $H_{S/J}(t) = P_Z(t) = H_Z(t)$ but $H_{S/J}(\rho_X) \neq P_Z(\rho_X) = H_Z(\rho_X)$. Since $J \subseteq J^{\text{sat}}$, it follows that, for every $t \geq \rho_X + 1$, $J_t = J_t^{\text{sat}}$, meanwhile $J_{\rho_X} \neq J_{\rho_X}^{\text{sat}}$. Thus $\text{sat}(J) = \rho_X + 1$.

If $\rho_X + 1 = \text{reg}(Z) - 1$, with analogous arguments as above it holds $\text{sat}(J) \leq \rho_X + 1 = \rho_Z$ and so $\text{reg}(X) = \text{reg}(Z)$.

If $\rho_X + 1 < \text{reg}(Z) - 1$, similarly we have $H_{S/J}(\rho_Z - 1) = P_Z(\rho_Z - 1)$ while $H_Z(\rho_Z - 1) \neq P_Z(\rho_Z - 1)$ and $H_{S/J}(\rho_Z) = P_Z(\rho_Z) = H_Z(\rho_Z)$, so $\text{sat}(J) = \rho_Z$. $\square$

Remark 2.7 Note that for an aCM curve $C \subset \mathbf{P}_K^n$ we have $\text{reg}(C) = \rho_C + 2 = \text{reg}(Z)$. If $\mathcal{L}$ is a non-aCM even liaison class of curves in $\mathbf{P}_K^3$ and C_0 is a minimal curve in $\mathcal{L}$, then, by [15, Lemma 2.2], it follows that $\text{reg}(C_0) = \rho_{C_0} + 1$. For a curve $C \subset \mathbf{P}_K^3$ with equal cohomology, in the sense of [15, Definition 2.1], the regularity is $\text{reg}(C) \geq \rho_C + 3$, thus $\text{reg}(C) = \text{reg}(Z)$.

Remark 2.8 Note that, for a connected curve $C \subset \mathbf{P}_K^n$ which is non-special, i.e. $h^1(\mathcal{O}_C(t)) = 0$ for every $t \geq 1$ (for example, any smooth rational curve):

- (a) If $\rho_C = 0$, then $\text{reg}(C) = 2$ and C is a rational normal curve.
- (b) If $\rho_C = 1$, then $\text{reg}(C) = 3$, $\deg(C) = n + g$ and C is not a rational smooth curve.
- (c) If $\rho_C \geq 2$, then $\text{reg}(C) = \rho_C + 1$.

In fact, $h^2(\mathcal{I}_C(t)) = 0$ for every $t \geq 1$ and $h^1(\mathcal{I}_C(t)) = h^2(\mathcal{I}_C(t))$ for every $t \geq \rho_C$. Then, if $\rho_C \geq 2$, we have $\text{reg}(C) = \rho_C + 1 \geq 3$. Otherwise, if $\rho_C = 0$, then $P_C(0) = 1 - g = H_C(0) = 1$ and $P_C(1) = \deg(C) + 1 - g = H_C(1) = n + 1$, so C is a rational normal curve and $\text{reg}(C) = 2$. Note that, for a 2-regular curve C we get $h^2(\mathcal{I}_C) = h^1(\mathcal{O}_C) = 0$ and, so, $P_C(0) = 1 - g = H_C(0) + h^1(\mathcal{I}_C) = 1 + h^1(\mathcal{I}_C)$. Thus, we have $h^1(\mathcal{I}_C) = -g$, that is possible only for $g = 0$. If $\rho_C = 1$, $\deg(C) = n + g$ and $g \neq 0$, thus $\text{reg}(C) = 3$.

Remark 2.9 (a) In [1], for an integral curve $C \subset \mathbf{P}_K^n$ of degree $\deg(C)$, Ballico shows that $\rho_Z \leq \left\lceil \frac{\deg(C)-1}{n-1} \right\rceil$.

(b) If $X \subset \mathbf{P}_K^n$ is any k -equidimensional ICM not c.i. scheme and W is the 0-dimensional scheme obtained from X by k successive general hyperplane sections, then [19, Proposition 4.3] gives

$$\text{reg}(X) \leq 1 + \max \{t \mid H^1(\mathcal{I}_W(t)) \neq 0\} + \max \{k(X) \dim(X), 1\}, \quad (2.3)$$

where $k(X)$ is the so-called Ellia-Migliore-Miró Roig number of X , i.e. the minimal nonnegative integer k such that $(x_0, \dots, x_n)^k$ annihilates the deficiency modules $M^i(X) := \bigoplus H^i(\mathcal{I}_X(t))$ for every integer i such that $1 \leq i \leq \dim(X)$. So the following bound for the regularity of a nondegenerate irreducible reduced projective variety X holds (see [19, 18, Theorem 1.3] with the references therein):

$$\text{reg}(X) \leq \left\lceil \frac{\deg(X) - 1}{\text{codim}(X)} \right\rceil + \max \{k(X) \dim(X), 1\}. \quad (2.4)$$

For an integral curve C such that $k(C) > 1$ it is evident that, if this bound is achieved, then $\text{reg}(C) = \rho_C + 1$. In [18, Theorem 1.5] Miyazaki proves that for any given positive integers c and k , with $c \geq k$, there exists a nondegenerate irreducible smooth projective curve C in $\mathbf{P}_K^{c+1}$ achieving the bound, with $k = k(C)$.

(c) Note that, for any ICM curve C , formula (2.3) implies that, if $\rho_C \geq \rho_Z$, then $\rho_C - \rho_Z < k(C)$. If $\rho_Z > \rho_C$, it is trivial that $\rho_Z - \rho_C < \deg(C)$, since $\rho_Z < \deg(C)$.

3. Regularity bounds by minimal generators and Hilbert function

By applying Lemma 2.6 to a curve $C \subset \mathbf{P}_K^n$ with general hyperplane section Z , we get $\text{reg}(C) = \max \{\rho_C + 1, \rho_Z + 1\} = \min \{t \geq \rho_Z + 1 \mid \Delta H_C(t) = \deg(C)\}$, being $\text{reg}(Z) = \rho_Z + 1$ by Remark 2.5(a). So, bounds for zero-dimensional schemes' regularity enter into our aims.

Recalling that, given a k -dimensional scheme $X \subset \mathbf{P}_K^n$, there exist positive integers $\beta_1 \leq \dots \leq \beta_{n-k}$ and a c.i. of type $(\beta_1, \dots, \beta_{n-k})$ containing X (see [13, Theorem 3.14 of Chapter VI]), we set the following definition.

DEFINITION 3.1 If C is a space curve and Y is a c.i. of type (β_1, β_2) containing C with β_1 and β_2 minimal, we let $B(C) = \beta_1 + \beta_2 - 1$.

The following statement is a particular case of [8, Theorem 3(c)] about Hilbert function under liaisons. We refer to [16] for definitions and results about liaisons.

Lemma 3.2

Let Y be a 0-dimensional c.i. containing properly a scheme Z defined by a saturated ideal $I(Z)$. Let Z' be the scheme algebraically linked to Z by Y , i.e. defined by $I(Z') = (I(Y) : I(Z))$. Let α and α' be, respectively, the initial degrees of $I(Z)/I(Y)$ and of $I(Z')/I(Y)$. Hence: $\text{reg}(Z) + \alpha' = \text{reg}(Z') + \alpha = \text{reg}(Y)$.

Remark 3.3 With the notation of the above lemma, note that $\alpha_{I(Z)} \leq \alpha$ and $\alpha_{I(Z')} \leq \alpha'$. In particular, let $C \subset \mathbf{P}_K^3$ be the curve algebraically linked to a plane curve C' of degree 4 by a c.i. of type $(3, 6)$. Being the aCM property preserved by liaison, C is an aCM curve and we can apply Lemma 3.2 with $\text{reg}(Y) = 3 + 6 - 1 = 8$, $\alpha' = 1$ and $\text{reg}(Z') = 4$, getting: $\alpha = \text{reg}(Y) - \text{reg}(Z') = 4$ and $\text{reg}(Z) = \text{reg}(Y) - \alpha' = 7$. Since we chose a c.i. of type $(3, 6)$, there exists a degree 3 curve through Z , thus $\alpha_{I(Z)} < 4 = \alpha$.

Proposition 3.4

Let $\beta_1 \leq \dots \leq \beta_{n-1}$ be degrees for which there exists a c.i. Y of type $(\beta_1, \dots, \beta_{n-1})$ containing C . So

- (i) $\text{reg}(C) \leq \max \{ \rho_C + 1, \sum_{i=1}^{n-1} \beta_i - n + 2 \}$.
- (ii) if Z is not a c.i. of type $(\beta_1, \dots, \beta_{n-1})$ (e.g. if $\deg(C) < \prod_{i=1}^{n-1} \beta_i$), then $\text{reg}(C) \leq \max \{ \rho_C + 1, \sum_{i=1}^{n-1} \beta_i - n + 1 \}$;
- (iii) the bound of (ii) is sharp if C is algebraically linked by Y to a curve having degenerate general hyperplane section;
- (iv) if C is a non-degenerate curve with $\text{reg}(C) = \sum_{i=1}^{n-1} \beta_i - n + 1 > \rho_C + 1$, then C is algebraically linked by Y to a curve C' with degenerate general hyperplane section.

Proof. All the statements are consequences of Lemmas 2.6 and 3.2, because regular sequences and linkages are preserved under hypersurface section. The equality $\text{reg}(Z) = \sum_{i=1}^{n-1} \beta_i - n + 2$ holds if and only if Z is a c.i. of type $(\beta_1, \dots, \beta_{n-1})$. $\square$

Remark 3.5 By formula (2.3), the following bound analogous to formula (2.4) can be deduced for a lCM projective scheme X which is not a c.i.

$$\text{reg}(X) \leq \sum_{i=1}^{n-1} \beta_i - n + \max \{ k(X) \dim(X), 1 \}. \quad (3.1)$$

Corollary 3.6

If $C \subset \mathbf{P}_K^3$ is integral and $f_1, \dots, f_s$ are minimal generators of I with degrees $d_1 \leq \dots \leq d_s$, then Proposition 3.4 holds with $\beta_1 = d_1$, $\beta_2 = d_2$.

Proof. Being C integral, minimal generators of I are always irreducible so that f_i, f_j form an S -regular sequence for every integers i, j such that $1 \leq i < j \leq s$. $\square$

Corollary 3.7

If $C \subset \mathbf{P}_K^3$ is an integral curve with $\text{reg}(C) = B(C) - j > \rho_C + 1$ and $\beta_1 \cdot \beta_2 - \deg(C) > j^2 + 1$, then C is algebraically linked to a curve the defining ideal of which has initial degree $\leq j$.

Proof. By recalling that linkage is preserved under hypersurface section, this statement follows by an application of [22, Laudal's lemma], being $\beta_1 \cdot \beta_2 - \deg(C)$ the degree of the linked curve and $\text{reg}(Z) = B(C) - j$. $\square$

Remark 3.8 In [10] Harris shows that the space curves C of degree d lying on an irreducible surface S of degree k and having extremal genus are exactly the smooth curves residual to a plane curve in a complete intersection of S with a surface of degree $n = \left\lceil \frac{d-1}{k} \right\rceil + 1$. Hence, since C is aCM, we have $\text{reg}(C) = \rho_C + 2 = \text{reg}(Z) = k + \left\lceil \frac{d-1}{k} \right\rceil - 1$.

The problem to recognize a curve with degenerate general hyperplane section has been studied by many authors and we use some answers to this question to investigate the condition $\text{reg}(C) = B(C) - 1 > \rho_C + 1$ for a curve $C \subset \mathbf{P}_K^3$.

Theorem 3.9

In characteristic 0, a non-degenerate curve $C \subset \mathbf{P}_K^3$, with $\text{reg}(C) > \rho_C + 1$ and $B(C) = \beta_1 + \beta_2 - 1$, is algebraically linked to a plane curve C' by a c.i. Y of type (β_1, β_2) if and only if $\text{reg}(C) = B(C) - 1$.

Proof. If C is algebraically linked to a plane curve, then the thesis follows by Proposition 3.4(iii).

Vice versa, since we have that $\rho_C + 1 < \text{reg}(C) = B(C) - 1$, then $\text{reg}(Z) = \text{reg}(C) = B(C) - 1$ and, hence, C is algebraically linked to a curve $C' \subset \mathbf{P}_K^3$ with degenerate general hyperplane section Z' . Thus, C' is either a plane curve or a double line or a pair of skew lines (see [12, Theorem 2.1]). So, if C' is not a plane curve, one has in particular that $\deg(C') = 2$ and the arithmetic genus of C' is $p_a(C') = -d$, with $d \geq 1$. Moreover, it is well known that $M^1(C) = M^1(C')^*(4 - \beta_1 - \beta_2)$ (see e.g. [16, Theorem 5.3.1]). Since $M^1(C')^*_t = \text{Hom}_K(M^1(C')_{-t}, K)$ and the minimum degree at which $M^1(C')$ does not vanish is $-d + 1$ (see e.g. [16, Example 1.5.10]), then the maximum degree at which $M^1(C)$ does not vanish is $\beta_1 + \beta_2 - 4 + d - 1$.

First of all, consider the case $d = 1$, in which C' is Buchsbaum (i.e. $k(C') = 1$) and $K \simeq M^1(C') = H^1(\mathcal{I}_{C'}(0))$. Thus, $K \simeq M^1(C) = H^1(\mathcal{I}_C(\beta_1 + \beta_2 - 4))$ and, for a general linear form h , one gets $(0 :_{M^1(C)} h) = M^1(C)$. By [16, Proposition 2.1.2(b)], it follows that $M^1(C)(-1) \simeq J^{\text{sat}}/J$, from which we obtain $\text{sat}(J) = \beta_1 + \beta_2 - 2 = \text{reg}(C)$. But, by Lemma 2.6, the equality $\text{sat}(J) = \text{reg}(C)$ occurs only when $\text{reg}(C) = \rho_C + 1$.

In the case $d \geq 2$, since the maximum degree at which $M^1(C)$ does not vanish is $\beta_1 + \beta_2 - 4 + d - 1$, we have $\text{reg}(C) \geq \beta_1 + \beta_2 - 4 + d - 1 + 2 \geq \beta_1 + \beta_2 - 1$, which cannot happen. $\square$

Remark 3.10 (a) Note that the curves satisfying Theorem 3.9 belong to the same liaison class since there is only one liaison class consisting of aCM projective schemes of codimension two (see e.g. [16, Corollary 6.1.4]).

(b) In the hypotheses of Theorem 3.9, we have $\text{reg}(Z') = \deg(C')$. Hence, by Lemma 3.2, we get $\beta_2 \geq \deg(C') + 1$. One could have expected that $\beta_2 \geq \deg(C')$, but the case $\beta_2 = \deg(C')$ is excluded by the hypotheses of Theorem 3.9 itself, since a curve C algebraically linked to a plane curve C' by a c.i. of type $(\beta_1, \deg(C'))$ is a c.i. of initial degree $= \beta_1 - 1 < \beta_1$ (see e.g. [16, Example 1.5.7(3)]).

(c) If $Y \subset \mathbf{P}_K^2$ is a c.i. of type (β_1, β_2) , then

$$\Delta H_Y : 1, 2, 3, \dots, \beta_1 - 1, \beta_1, \beta_1, \dots, \beta_1, \beta_1, \beta_1 - 1, \dots, 3, 2, 1$$

where β_1 appears $n = \beta_2 - \beta_1 + 1$ times. If $Z' \subset \mathbf{P}_K^2$ is a degenerate 0-dimensional scheme of degree $\bar{d}$, then $\Delta H_{Z'} : 1, 1, 1, \dots, 1, 1, 0$, where 1 appears $\bar{d}$ times. Hence, by [8, Theorem 3], if Z is the scheme algebraically linked to Z' by Y , we get

$$\Delta H_Z(t) = \begin{cases} \Delta H_Y(t), & 0 \leq t \leq \beta_1 + \beta_2 - 2 - \bar{d} \\ \Delta H_Y(t) - 1, & \beta_1 + \beta_2 - 1 - \bar{d} \leq t < \beta_1 + \beta_2 - 2 \\ 0, & \beta_1 + \beta_2 - 2 \leq t \end{cases}.$$

Note that, $\Delta H_Z(t)$ is strictly decreasing if, and only if, $\bar{d} \leq \beta_1$ or $\bar{d} \geq \beta_1 + n - 1 = \beta_2$. In these cases, there exists an irreducible smooth aCM curve whose generic plane section has H_Z as Hilbert function [14]. So, if C is an integral curve satisfying Theorem 3.9, then $\deg(C') \leq \beta_1$, since $\deg(C') \leq \beta_2 - 1 < \beta_2$.

For integral space curves over a field of characteristic 0 one can say something more about $\rho_Z = \text{reg}(Z) - 1$ basing on the shape of Borel-fixed ideals. Namely, it is known that (see, for example, [9]) the Borel-fixed ideal $\text{gin}(I(Z))$ is of the following type

$$(x_0^s, x_0^{s-1}x_1^{\lambda_{s-1}}, \dots, x_0x_1^{\lambda_1}, x_1^{\lambda_0}),$$

where $\lambda_0 = \rho_Z + 1$, $\lambda_i - 2 \leq \lambda_{i+1} \leq \lambda_i - 1$, for $0 \leq i \leq s-1$, $s = \alpha_{I(Z)}$, $\deg(Z) = \sum_{i=0}^{s-1} \lambda_i$ and ([9, Corollary 4.9])

$$\frac{\deg(Z)}{s} + \frac{s-1}{2} \leq \lambda_0 \leq \frac{\deg(Z)}{s} + s - 1. \quad (3.2)$$

Proposition 3.11

Let $\gamma_C := \max \{t \in \mathbf{N} : \deg(C) > t^2 + 1\}$, then

- (a) $\deg(C) > \alpha_I^2 + 1 \Rightarrow \text{reg}(C) \leq \max \{ \rho_C + 1, \lfloor \frac{\deg(C)}{\alpha_I} \rfloor + \alpha_I - 1 \}$.
- (b) $\deg(C) \leq \alpha_I^2 + 1 \Rightarrow \text{reg}(C) \leq \max \{ \rho_C + 1, \lfloor \frac{\deg(C)}{\gamma_C + 1} \rfloor + \gamma_C \}$.

Proof. (a) By Lemma 2.6 the thesis is an easy consequence of formula (3.2) and [22, Laudal's lemma], because one obtains that $s = \alpha_{\bar{J}_{\text{sat}}} = \alpha_J = \alpha_I$.

(b) In this case s is not known. However we obtain that $s > \gamma_C$, namely, in our hypothesis it can happen that either $s = \alpha_I > \gamma_C$ or $2 \leq s < \alpha_I$. In the second case, by [22] $\deg(C) \leq s^2 + 1$ and so $s > \gamma_C$. Therefore $\deg(C) \geq \lambda_0 + \lambda_1 + \dots + \lambda_{\gamma_C} \geq \lambda_0 + \lambda_0 - 2 + \dots + \lambda_0 - 2\gamma_C = (\gamma_C + 1)(\lambda_0 - \gamma_C)$ and we are done. $\square$

4. Examples

We provide examples of curves to test the sharpness of the regularity bounds of Section 3. We apply the technique used in [15] to construct curves with “high degree generators” involving basic double linkages. For the sake of completeness, we recall briefly the procedure to obtain a basic double link of type $(a_1, \dots, a_{n-1})$ from a curve C of $\mathbf{P}_K^n$. Consider homogeneous polynomials $F_1, \dots, F_{n-1}$ of $K[x_0, \dots, x_n]$ of degrees $a_1, \dots, a_{n-1}$ respectively, with F_1 general, $F_2, \dots, F_{n-1}$ belonging to I and $(F_1, \dots, F_{n-1})$ a regular sequence, then the curve $C' \subset \mathbf{P}_K^n$ defined by the saturated ideal $(F_1I, F_2, \dots, F_{n-1})$ is called *basic double link*.

In order to obtain interesting examples of integral curves in $\mathbf{P}_K^3$, we apply also the deformation technique with constant cohomology described in [20, Proposition 3.5]. More precisely, let $C \subset \mathbf{P}_K^3$ be an integral curve with β_1, β_2 as in Proposition 3.4 and $e(C) = \max \{t \in \mathbb{N} \mid h^2(\mathcal{I}_C(t)) \neq 0\}$ its speciality index. Then, the curve obtained from C by a basic double link of type (w, d) , with $d \leq e(C) + 4 + w$ such that $d = \beta_1$ or $d \geq \beta_2$, can be deformed with constant cohomology to an integral curve. In this case we do not compute the curve explicitly and, so, we do not have further information about the hyperplane section Z .

EXAMPLE 4.1 Here are examples of curves for every case of Lemma 2.6.

(1) Let $C \subset \mathbf{P}_K^4$ be the rational curve of degree 30 parametrized by

$$\begin{cases} x_0 = u^{30} + v^{30} \\ x_1 = u^{29}v + u^{19}v^{11} + u^9v^{21} \\ x_2 = u^{29}v + u^{18}v^{12} + u^8v^{22} \\ x_3 = u^{27}v^3 + u^{17}v^{13} + u^7v^{23} \\ x_4 = u^{26}v^4 + u^{16}v^{14} + u^6v^{24} \end{cases}.$$

In characteristic 31991 we get $\rho_C + 1 = 21$. Since $\rho_Z \leq \lceil \frac{\deg(C)-1}{n-1} \rceil = 10$, it is $\text{reg}(C) = 21$ by Lemma 2.6(1). Direct computations tell us that $\rho_Z = 4$.

(2) Lemma 2.6(2) holds not only for aCM curves. Let $C_0 \subset \mathbf{P}_K^3$ be a general elliptic curve of degree 5. By applying a sequence of two basic double links to C_0 , respectively of types $(1, 5)$ and $(1, 7)$, we obtain a non-aCM curve X of degree 17 with $\text{reg}(X) = 7 = \rho_X + 2$, in characteristics 31991 and 0. In fact, $\text{reg}(Z) = 7$.

(3) Let $C_0 \subset \mathbf{P}_K^3$ be a general rational curve of degree 4. By applying a sequence of two basic double links to C_0 , respectively of types $(1, 4)$ and $(1, 6)$, we obtain a curve X of degree 14 and $\text{reg}(X) = 6 = \rho_X + 3$, in characteristics 31991 and 0. In fact, $\text{reg}(Z) = 6$.

EXAMPLE 4.2 For every case of Lemma 2.6, we give examples of curves in $\mathbf{P}_K^4$ for which the bound of Proposition 3.4(ii) is sharp.

(1) For a general degree 6 *elliptic* curve C we get $\rho_C = 2$, $\text{reg}(C) = 3$, $\beta_1 = \beta_2 = \beta_3 = 2$ and, so, $\text{reg}(C) = \beta_1 + \beta_2 + \beta_3 - 3$.

(2) For a general degree 5 *elliptic* curve C (which is aCM) we get $\rho_C = 1$, $\text{reg}(C) = 3$, $\beta_1 = \beta_2 = \beta_3 = 2$ and, so, $\text{reg}(C) = \beta_1 + \beta_2 + \beta_3 - 3 = \rho_C + 2$. Besides, let C_0 be a space general *rational* curve of degree 5 embedded in $\mathbf{P}_K^4$ and C_1 be the not aCM curve algebraically linked to C_0 by a c.i. of type $(\beta_1, \beta_2, \beta_3) = (2, 3, 3)$. Thus $\rho_{C_1} + 2 = \text{reg}(C_1) = 5 = \beta_1 + \beta_2 + \beta_3 - 3$.

(3) Let C_0 be a *space elliptic* curve of degree 5 embedded in $\mathbf{P}_K^4$. If C is the curve algebraically linked to C_0 by a c.i. of type $(\beta_1, \beta_2, \beta_3) = (2, 3, 3)$, then $\rho_C + 3 = \text{reg}(C) = 5 = \beta_1 + \beta_2 + \beta_3 - 3$.

EXAMPLE 4.3 In this example we point out that the hypothesis “integral” in Corollary 3.6 is essential. If we apply a basic double link of type $(1, 8)$ to a general elliptic curve of degree 5, we get a non-integral curve C of degree 13, the ideal of which is generated by 5 polynomials of degree 4 and by 1 polynomial of degree 8, with $\rho_C = 6$, but $\max \{\rho_C + 1, d_1 + d_2 - 1\} = 7 < 8 = \text{reg}(C)$.

EXAMPLE 4.4 Here is a family of integral curves $\{C_w\}_w$ in $\mathbf{P}_K^3$ for which the bound of Proposition 3.4(ii) is sharp with $\text{reg}(C_w) = \rho_{C_w} + 1 = B(C_w) - 1$. Let $C_0 \subset \mathbf{P}_K^3$ be the following smooth rational curve

$$\begin{cases} x_0 = u^5 + v^5 \\ x_1 = u^4v + uv^4 \\ x_2 = u^3v^2 + v^5 \\ x_3 = u^2v^3 \end{cases} ,$$

with Hilbert function $H : 1, 4, 9, 5t + 1$. Its ideal $I(C_0)$ is generated by 1 polynomial of degree 2 and by 4 polynomials of degree 4. Then $4 \leq \text{reg}(C_0) \leq \max\{4, 2 + 4 - 2\}$, indeed $\text{reg}(C_0) = 4$. By [20, Proposition 3.5], any basic double link of C_0 of type $(w, 4)$, with $w \geq 1$, can be deformed with constant cohomology in an integral curve C_w such that $\deg(C_w) = 5 + 4w$, $\text{reg}(C_w) = 4 + w$, $\rho_{C_w} = 3 + w$. Moreover, if $w = 1$ the initial degree d_1 of $I(C_w)$ is 3 and $d_2 = 4$, otherwise if $w > 1$ then $d_1 = 4$ and $d_2 = 2 + w$, so that $\max\{\rho_{C_w} + 1, d_1 + d_2 - 2\} = 4 + w$.

EXAMPLE 4.5 The bounds of Proposition 3.11 are sharp for the following integral curves in $\mathbf{P}_K^3$. For the curves described in (1) and in (2) Corollary 3.7 holds.

(1) By [20, Proposition 3.5], the curve X of Example 4.1(2) can be deformed with constant cohomology to an integral curve C of degree 17, regularity $\text{reg}(C) = 7 = \rho_C + 2$, initial degree $\alpha_I = \beta_1 = 5 = \beta_2$. Thus $\gamma_C = 3$ and $\text{reg}(C) = 7 = \max\{\rho_C + 1, \lfloor \frac{\deg(C)}{4} \rfloor + 3\}$, whereas $\max\{\rho_C + 1, \beta_1 + \beta_2 - 2\} = 8$ and $\max\{\rho_C + 1, \lceil \frac{\deg(C)-1}{2} \rceil + 1\} = 9$. By Corollary 3.7 and Theorem 3.9, C is algebraically linked to a curve of initial degree 2.

(2) By applying a further basic double link of type $(1, 4)$ to the curve X of Example 4.1(3) we obtain a curve which, by [20, Proposition 3.5], can be deformed with constant cohomology to an integral curve C of degree 18, regularity $\text{reg}(C) = 7 = \rho_C + 3$, initial degree $\beta_1 = 4$ and $\beta_2 = 6$. Thus $\text{reg}(C) = 7 = \max\{\rho_C + 1, \lfloor \frac{\deg(C)}{\beta_1} \rfloor + \beta_1 - 1\}$, whereas $\max\{\rho_C + 1, \beta_1 + \beta_2 - 2\} = 8$ and $\max\{\rho_C + 1, \lceil \frac{\deg(C)-1}{2} \rceil + 1\} = 10$. By Corollary 3.7 and Theorem 3.9, C is algebraically linked to a curve of initial degree 2.

(3) In [17, Example 3.2], over a field of characteristic 0, a family $\{Y_m\}_{m \geq 0}$ of 2-Buchsbaum smooth integral curves is constructed such that $\text{reg}(Y_m) = \max\{m + 4, 2m + 3\}$, the initial degree is $\alpha_m = \beta_1 = \beta_2 = m + 3$, $\deg(Y_m) = m^2 + 4m + 6 < \alpha_m^2 + 1$ and $\gamma_m := \gamma_{Y_m} = m + 2$.

If $m = 0$, Y_0 is a general elliptic curve of degree 6 and $4 = \text{reg}(Y_0) = \rho_{Y_0} + 1$.

If $m = 1$, then $5 = \rho_{Y_1} + 1 = \text{reg}(Y_1) = \max\{\rho_{Y_1} + 1, \lfloor \frac{\deg(Y_1)}{4} \rfloor + 3\} < 6 = \max\{\rho_{Y_1} + 1, \beta_1 + \beta_2 - 2\} < 8 = \max\{\rho_{Y_1} + 1, \lceil \frac{\deg(Y_1)-1}{2} \rceil + 1\}$.

If $m \geq 2$, then $\rho_{Y_m} + 2 = \text{reg}(Y_m) = 2m + 3 = \max\{\rho_{Y_m} + 1, \lfloor \frac{\deg(Y_m)}{m+3} \rfloor + m + 2\} < 2m + 4 = \max\{\rho_{Y_m} + 1, \beta_1 + \beta_2 - 2\} < \max\{\rho_{Y_m} + 1, \lceil \frac{\deg(Y_m)-1}{2} \rceil + 1\}$.

EXAMPLE 4.6 By the following examples we note that the hyperplane section Z' of a curve C' algebraically linked to a space curves C with $\text{reg}(C) \geq B(C) - 1$ can be either degenerate or non-degenerate.

(1) As it is shown in the proof of Theorem 3.9, every space curve C linked to a double line of arithmetic genus $-d \leq -2$ has regularity $\text{reg}(C) = \rho_C + 1 \geq B(C) - 1$, being $\text{reg}(C) - (B(C) - 1) = d - 2$.

(2) Let C_0 the rational curve of Example 4.4, with $\text{reg}(C_0) = \rho_{C_0} + 1 = B(C_0) - 1$. We compute that the general hyperplane section Z of C_0 has regularity 3 and defining ideal generated by 1 polynomial of degree 2 and by 2 polynomials of degree 3. If Z' is algebraically linked to Z by a c.i. of type $(2, 4)$, then Z' is *non-degenerate*, otherwise $\text{reg}(Z') = \deg(Z') = 2 \cdot 4 - 5 = 3$ and, by Lemma 3.2, we would get $\alpha = \text{reg}(Y) - \text{reg}(Z') = 2$, meanwhile we have $\alpha = 3$ by construction.

(3) Let $C \subset \mathbf{P}_K^3$ be the rational curve

$$\begin{cases} x_0 = u^9 + v^9 \\ x_1 = u^8v + uv^8 \\ x_2 = u^7v^2 + u^2v^7 \\ x_3 = u^6v^3 \end{cases} ,$$

with Hilbert function $H : 1, 4, 10, 20, 31, 41, 9t - 3$. Its defining ideal $I(C)$ is generated by 4 polynomial of degree 4, 1 polynomial of degree 5 and 1 polynomial of degree 7. Hence, $B(C) = 4 + 4 - 1 = 7$ and $\text{reg}(C) = \rho_C + 1 = 7 > B(C) - 1 = 6$. If C' is linked to C by a c.i. of minimum type $(4, 4)$, then $\deg(C') = 16 - 9 = 7$ and, hence, its general hyperplane section Z' is *non-degenerate*, otherwise $\text{reg}(Z') = 7$. Indeed, since $\alpha > 0$, by Lemma 3.2 we get $\text{reg}(Z') = \text{reg}(Y) - \alpha = 7 - \alpha < 7$.

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