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Notes on the Diederich-Sukhov-Tumanov normalization for almost complex structures

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Abstract

In this paper, written for non specialists, we discuss several points in the elementary theory of almost complex manifolds, with a focus on the question of choice of special coordinates and on the obstruction given by the Nijenhuis tensor.

1. Introduction

1.1. My motivation for writing these notes came first from my own desire of getting a better (more geometric) understanding of the Diederich-Sukhov-Tumanov normalization (see 1.2.) of an almost complex structure along a J-holomorphic disc. In particular, I wanted to have more direct discussions in terms of the tensor J itself, although this failed to simplify the crucial proofs.

The proofs given here for the normalizations, somewhat differ from the original ones. I completely avoided what I consider to be notationally tedious formulas for coordinate changes by applying Proposition 2, i.e. by basing the proof on a formula for the Laplacian of J-holomorphic discs. The Diederich-Sukhov normalization is a normalization just at a point, and that is a simple matter, at the level of Taylor expansion. The Sukhov-Tumanov normalization is along a disc and this is more subtle, since it seems to require some non-local work. I made a special effort for clarification of the proof by completely separating two steps in the Sukhov-Tumanov normalization: a first step in which a differential equation has to be solved and a second elementary step completely analogous to the Diederich-Sukhov case. I am very grateful to A. Tumanov

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for explaining to me how the normalization [11, Lemma 2.2] is not only for almost complex structures close to the standard one (which was not clear to me), using the result explained in the appendix. In this paper, an effort has been made to make smoothness requirements clear and somewhat minimal. In particular, smoothness requirements are carefully stated in Theorem 2, but they were left as they came naturally in the proof without searching for improvement.

The Diederich-Sukhov-Tumanov normalization has been used very efficiently in [10, 11] and it is my feeling that it may be helpful in questions related to pluripolarity (but this is only related to work in progress).

The heart of these notes is Section 3, but I took the opportunity provided by writing these notes to gather facts in the very elementary theory of almost complex structures. Although most of these facts have been well known for a long time, they seem to be usually skipped in expositions. Some other facts are less known but may be too quickly treated in papers. So, I felt that there was still some lack of an easy reference. These notes are written having in mind a reader who is absolutely not a specialist in almost complex analysis. They are at the opposite of notes where 'details are left to the reader' (especially when the details are tedious). This explains their length. Repetitions have been intentionally made in order to avoid (existing) risks of misreading definitions. The short appendix is in the spirit of an elementary self contained paper.

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1.2. Let us just have a short preliminary discussion of the Diederich-Sukhov-Tumanov normalization (with explanations given later). It is standard that if J is an almost complex structure defined near 0 in $\mathbf{R}^{2n}$, after a change of variable one can assume that $J(0)$ is the standard complex structure J_{st} given by identifying $\mathbf{R}^{2n}$ and $\mathbf{C}^n$, with the complex structure corresponding to multiplication by i. This is just the elementary algebraic fact that any endomorphism T of $\mathbf{R}^{2n}$, that satisfies $T^2 = -\mathbf{1}$, corresponds to multiplication by i in appropriate complex coordinates. J-holomorphic curves i.e. maps from an open set in $\mathbf{C}$ into $(\mathbf{R}^{2n}, J)$ are defined by the equation

$$\frac{\partial u}{\partial y} = J(u)\frac{\partial u}{\partial x} \ .$$

It happens that (if J is close enough to J_{st}) this equation can be re-written as:

$$\frac{\partial \overline{u}}{\partial \zeta} = Q(u)\frac{\partial u}{\partial \zeta} \ ,$$

where Q is $\mathbf{C}$-linear, not only $\mathbf{R}$-linear (so given by an $(n\times n)$ complex matrix). $J = J_{st}$ corresponds to $Q = 0$, Diederich and Sukhov showed that one can chose coordinates so that $J(0) = J_{st}$, i.e. $Q(0) = 0$, and $Q_{\overline{z}}(0) = 0$, and they pointed out that $Q_z(0) = 0$ cannot be achieved in general. Sukhov and Tumanov did the same normalization along a J-holomorphic disc.

2. Basic notions and notations. The Diederich-Sukhov normalization

2.1) *The operators* J *and* $\overline{Q}$

In $\mathbf{R}^{2n}$, coordinates will be denoted by $(x_1, y_1, \ldots, x_n, y_n)$ and $\mathbf{R}^{2n}$ is identified with $\mathbf{C}^n$, in which the variable will be denoted by $z = (z_1, \ldots, z_n)$, $z_j = x_j + y_j$. An almost complex structure J on some open set $\Omega \subset \mathbf{R}^{2n}$ is the data at each point $z \in \Omega$ of an endomorphism $J(z)$ of the tangent space to $\mathbf{R}^{2n}$ at p satisfying $J(p)^2 = -\mathbf{1}$. The standard complex structure given by multiplication by i corresponds in real notations to the $(2n \times 2n)$ matrix made of blocks

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

along the diagonal. If $p \mapsto J(p)$ is of class $\mathcal{C}^\alpha$, J is said to be of class $\mathcal{C}^\alpha$.

Let $\mathbf{D}$ be the unit disc in $\mathbf{C}$, in which the variable will always be denoted by ζ. However we will write $\zeta = x + iy$ (rather than $\xi + i\eta$ that would be logical). So, in what follows, we will always have z in $\mathbf{C}^n$, but x and y in $\mathbf{R}$. A J-holomorphic disc, in Ω, is a map u from $\mathbf{D}$ into Ω such that

$$\frac{\partial u}{\partial y} = J(u) \frac{\partial u}{\partial x}\ , \tag{$*$}$$

which means that du is $j_{st} - J$ linear, j_{st} denoting the standard complex structure on $\mathbf{C}$, for which $j_{st} \frac{\partial}{\partial x} = \frac{\partial}{\partial y}$. Long before Gromov made these discs an essential tool, Nijenhuis and Woolf established the basic theory. The main reference is [8], but more recent expositions can be found in several places including [9] that is very helpful, [4, 5] and [3] (where a special care was taken to give short and very elementary proofs). Here we shall not discuss basic facts such as: If the structure J is of class $\mathcal{C}^\alpha$ $(\alpha > 0)$, for any point $p \in \Omega$ and any tangent vector V at p, there exists a J-holomorphic disc u with $u(0) = p$ and $\frac{\partial u}{\partial x}(0) = \lambda V$, for some $\lambda > 0$, and all J-holomorphic disc are of class $\mathcal{C}^{\alpha+1}$ if $\alpha \notin \mathbf{N}$.

As it is classical, we set $\frac{\partial u}{\partial \zeta} = \frac{1}{2}(\frac{\partial u}{\partial x} - J_{st} \frac{\partial u}{\partial y})$, i.e. with complex ($\mathbf{C}^n$) notations: $\frac{\partial u}{\partial \zeta} = \frac{1}{2}(\frac{\partial u}{\partial x} - i \frac{\partial u}{\partial y})$. Similarly $\frac{\partial u}{\partial \overline{\zeta}} = \frac{1}{2}(\frac{\partial u}{\partial x} + J_{st} \frac{\partial u}{\partial y})$. One therefore has $\frac{\partial u}{\partial x} = \frac{\partial u}{\partial \zeta} + \frac{\partial u}{\partial \overline{\zeta}}$ and $J_{st} \frac{\partial u}{\partial y} = -\frac{\partial u}{\partial \zeta} + \frac{\partial u}{\partial \overline{\zeta}}$. Since $J_{st}^2 = -\mathbf{1}$, the equation $(*)$ for J-holomorphicity becomes:

$$J_{st} \frac{\partial u}{\partial \zeta} - J_{st} \frac{\partial u}{\partial \overline{\zeta}} = J\Big(\frac{\partial u}{\partial \zeta}\Big) + J\Big(\frac{\partial u}{\partial \overline{\zeta}}\Big)\ ,$$

where $J = J\big(u(\zeta)\big)$. So

$$[J + J_{st}] \frac{\partial u}{\partial \overline{\zeta}} = [J_{st} - J] \frac{\partial u}{\partial \zeta}\ .$$

If we restrict the case when $[J + J_{st}]$ is an invertible endomorphism of $\mathbf{R}^{2n}$ (which happens in particular if the operator norm of $J - J_{st}$ is < 1), one has the following equation from J-holomorphic discs:

$$\frac{\partial u}{\partial \overline{\zeta}} = \overline{Q}(u) \frac{\partial u}{\partial \zeta}\ ,$$

with $\overline{Q} \ = \ [J + J_{st}]^{-1}[J_{st} - J]$.

Now, (explaining the choice of notation,) it happens that $J^2 = -\mathbf{1}$ has the consequence that the operator $\overline{Q}$ is not an arbitrary endomorphism of $\mathbf{R}^{2n}$. Indeed $\overline{Q}$ is conjugate linear in the identification of $\mathbf{R}^{2n}$ and $\mathbf{C}^n$, i.e. in complex notations $\overline{Q}(iz) = -iQ(z)$, in real notations $\overline{Q} \circ J_{st} = -J_{st} \circ \overline{Q}$.

There is a completely obvious reason why $\overline{Q}$ has to be conjugate linear, namely that if $\zeta \mapsto u(\zeta)$ is J-holomorphic, so is $\zeta \mapsto u(\lambda\zeta)$, for any fixed complex number λ. The latter fact is of course completely linked to the property $J^2 = -\mathbf{1}$.

We however write an elementary algebraic checking.

Since $J^2 = J_{st}^2 = -\mathbf{1}$, $[J_{st} + J]J_{st} = J[J_{st} + J]$ $(= [-\mathbf{1} + JJ_{st}])$. Taking inverses: $-J_{st}[J_{st}+J]^{-1} = -[J_{st}+J]^{-1}J$. We also have $J[J_{st} - J] = -[J_{st} - J]J_{st}$ $(= [\mathbf{1} + JJ_{st}])$. Therefore:

$$\begin{aligned} J_{st} \circ \overline{Q} &= J_{st}[J + J_{st}]^{-1}[J_{st} - J] = [J + J_{st}]^{-1}J[J_{st} - J] \\ &= -[J + J_{st}]^{-1}[J_{st} - J]J_{st} = -\overline{Q}J_{st} \,, \end{aligned}$$

as claimed.

One has $[J + J_{st}]\overline{Q} = [J_{st} - J]$. One can solve for J, given $\overline{Q}$. Then, given a conjugate linear operator $\overline{Q}$, there corresponds an almost complex structure defined by J where J is given by

$$J = J_{st}[\mathbf{1} - \overline{Q}][\mathbf{1} + \overline{Q}]^{-1}.$$

This makes sense in particular when the operator norm of $\overline{Q}$ is < 1 (corresponding to J close to J_{st}). It is again an elementary algebraic fact that conjugate linearity of $\overline{Q}$ (which below is used in $[\mathbf{1} - \overline{Q}]J_{st} = J_{st}[\mathbf{1} + \overline{Q}]$), and $J_{st}^2 = -\mathbf{1}$, is enough to imply $J^2 = -\mathbf{1}$, as we now check.

$$\begin{aligned} J^2 &= J_{st}[\mathbf{1} - \overline{Q}][\mathbf{1} + \overline{Q}]^{-1}J_{st}[\mathbf{1} - \overline{Q}][\mathbf{1} + \overline{Q}]^{-1} \\ &= [\mathbf{1} + \overline{Q}]J_{st}[\mathbf{1} + \overline{Q}]^{-1}][\mathbf{1} + \overline{Q}]J_{st}[\mathbf{1} + \overline{Q}]^{-1} = -\mathbf{1} \,. \end{aligned}$$

The conclusion is that almost complex structures that are close to the standard one can equivalently be given either by the endomorphism J with $J^2 = -\mathbf{1}$, or by the conjugate linear operator $\overline{Q}$, that came in the equation of J-holomorphic discs.

Notes:

(i) This is not the only occurrence of conjugate linear operators in the theory. Indeed, consider an almost complex structure $J = J_{st} + \epsilon$, close to J_{st}. Then $J^2 = -\mathbf{1}$ yields $J_{st} \circ \epsilon + \epsilon \circ J_{st} + O(|\epsilon|^2) = 0$. So, at the infinitesimal level (i.e. for the Lie algebra) $J_{st} \circ \epsilon = -\epsilon \circ J_{st}$.

(ii) L. Lempert pointed out to me that the operator $\overline{Q}$ has a long history, in particular for integrable complex structures, being called the deformation tensor (see works by Kodaira and Morrow-Kodaira).

2.2) *The operator Q and z, $\overline{z}$ derivatives*

For any $\mathbf{R}$ endomorphism of $\mathbf{C}^n$, the conjugate operator $\overline{R}$ is defined by $\overline{R}(t) = \overline{R(t)}$ $(t \in \mathbf{C}^n)$. For z in say $\Omega \subset \mathbf{C}^n$, we have defined the conjugate linear operator $\overline{Q} = \overline{Q}(z)$. This is the conjugate of the $\mathbf{C}$-linear operator $Q = Q(z)$ defined by $[Q(z)](t) = \overline{[\overline{Q}(z)](t)}$. So we shall consider Q given, at each point z, by an $(n \times n)$ matrix with complex coefficients (instead of a $(2n \times 2n)$ real matrix). The equation for J-holomorphic disc can now be written entirely in complex notations

$$\frac{\partial \overline{u}}{\partial \zeta} = Q(u) \frac{\partial u}{\partial \zeta} \ . \tag{$**$}$$

In all that follows, when thinking in terms of matrix multilplication, one should of course treat $\frac{\partial u}{\partial \zeta}$ as a column vector although we will usually write $\frac{\partial u}{\partial \zeta}$ as a row vector with entries $\frac{\partial u_j}{\partial \zeta}$.

The meaning of the partial derivatives $\frac{\partial Q}{\partial z_j}$ and $\frac{\partial Q}{\partial \overline{z}_j}$ is obvious, one differentiates the $(n \times n)$ matrix representing Q coefficient-wise. This is why several results below are written in terms of Q. It is however better to take a more general approach that allows one in particular to differentiate J, and that is not simply differentiation coefficient-wise of a complex matrix. The space of $\mathbf{R}$-linear endomorphisms of $\mathbf{C}^n$ has of course a structure of complex vector space. For any $\mathbf{R}$-endomorphism R and any $\lambda = a+ib \in \mathbf{C}$, one defines $\lambda R = aR \ + \ J_{st} \circ (bR)$. (For a $\mathbf{C}$-linear endomorphism, represented by a $(n \times n)$ complex matrix instead of a $(2n \times 2n)$ real matrix, this is of course just multiplication of each coefficient of the complex matrix by λ.) Let $z \mapsto R(z)$ be a $\mathcal{C}^1$ map from an open set in $\mathbf{C}^n$ into the space of $\mathbf{R}$-endomorphisms of $\mathbf{C}^n$. Then one sets

$$\frac{\partial R}{\partial z_j} = \frac{1}{2}\Big(\frac{\partial R}{\partial x_j} - J_{st} \circ \frac{\partial R}{\partial y_j} \Big) \ ,$$

i.e. for fixed t,

$$\frac{\partial R}{\partial z_j}(t) = \frac{1}{2}\Big(\frac{\partial}{\partial x_j}([R(z)](t)) - i \frac{\partial}{\partial y_j}([R(z)](t)) \Big) \, .$$

In terms of coefficients of matrices, if $R(z)$ is represented, in the standard basis of $\mathbf{R}^{2n}$, by a real $(2n \times 2n)$ matrix $(r_{j,k}(z))$, then the endomorphism $\frac{\partial R}{\partial z_j}$ is represented by the real $(2n \times 2n)$ matrix $\rho_{j,k}(z)$ defined by the relations

$$\rho_{2p-1,k} + i\rho_{2p,k} = \frac{\partial}{\partial z_j}(r_{2p-1,k} + i r_{2p,k}) \quad (1 \le p \le n \, , \ 1 \le k \le 2n) \, .$$

If one mixes real and complex notations, in a way that it may be better to avoid (as the lines below may show), $R(z)$ an $\mathbf{R}$ linear map from $\mathbf{R}^{2n}$ into $\mathbf{C}^n$, can be represented by a rectangular $(n \times 2n)$ matrix with complex coefficients. Partial differentiation $\frac{\partial}{\partial z_j}$ is then again simply differentiation of each coefficient in the matrix. Some care is needed due to the lack of commutativity with J_{st}, although one has

$$\frac{\partial}{\partial x_j}(R_1 \circ R_2) = \frac{\partial R_1}{\partial x_j} \circ R_2 + R_1 \circ \frac{\partial R_2}{\partial x_j} \ ,$$

and similarly for $\frac{\partial}{\partial y_j}$, there is no such formula for $\frac{\partial}{\partial z_j}$ (see 2.4.3).

Similarly

$$\frac{\partial R}{\partial \overline{z}_j} = \frac{1}{2}\Big(\frac{\partial R}{\partial x_j} + J_{st} \circ \frac{\partial R}{\partial y_j} \Big) \, .$$

With the same notations, $\frac{\partial R}{\partial \overline{z}_j} = 0$ on some open set for every $j \in \{1, \dots, n\}$, therefore means that for every $p \in \{1, \dots, n\}$ and $k \in \{1, \dots, 2n\}$ the function $z \mapsto r_{2p-1,k}(z) + ir_{2p,k}(z)$ is holomorphic on that open set.It is straightforward to check that

$$\frac{\partial R}{\partial z_j} = \overline{\frac{\partial \overline{R}}{\partial \overline{z}_j}} \ , \text{ and}$$

$$\frac{\partial R}{\partial \overline{z}_j} = \overline{\frac{\partial \overline{R}}{\partial z_j}} \ .$$

Finally R_z (resp. $R_{\overline{z}}$) denotes the Frechet derivative that to each (tangent vector) $t \in \mathbf{C}^n$ associates the $\mathbf{R}$-endomorphism of $\mathbf{C}^n$ defined by

$$R_z.t = \sum_j t_j \frac{\partial R}{\partial z_j} \quad \Big(\text{resp. } R_{\overline{z}}.t = \sum_j t_j \frac{\partial R}{\partial \overline{z}_j}\Big) .$$

Note the $\mathbf{C}$-linearity in t. For $X \in \mathbf{C}^n$, $[R_z.t](X)$ therefore denotes the vector in $\mathbf{C}^n$ that is the image of X under the endomorphism $R_z.t$ (multiplication of the matrix representing $R_z.t$ and X written as a column vector). One has the first order Taylor expansion

$$R(z + \Delta z) \ = \ R(z) + R_z.\Delta z + R_{\overline{z}}.\overline{\Delta z} + o(|\Delta z|) \ ,$$

$R_z = 0$ (resp. $R_{\overline{z}} = 0$), at a given point z, is equivalent to having all the z_j (resp. $\overline{z}_j$) derivatives of R vanishing at that point.

Proposition 1

Let J be a $\mathcal{C}^1$ almost complex structure defined near p in $\mathbf{C}^n$. Assume that $J(p) = J_{st}$ (so $Q(p) = 0$).

(i) *The following are equivalent: $Q_z(p) = 0$, $\overline{Q}_{\overline{z}}(p) = 0$, $J_{\overline{z}}(p) = 0$.*

(ii) *The following are equivalent $Q_{\overline{z}}(p) = 0$, $\overline{Q}_z(p) = 0$, $J_z(p) = 0$.*

The first equivalence in (i) and (ii) is given by the lines above. Since $J = J_{st} \circ [\mathbf{1} - \overline{Q}][\mathbf{1} + \overline{Q}]^{-1}$ one has near p, $J = J_{st}[\mathbf{1} - 2\overline{Q}] + o(|Q|)$. Partial differentiation with respect to z_j and composition with J_{st} commute. More generally, for any fixed linear map A from $\mathbf{C}^n$ into $\mathbf{C}^n$ that is $\mathbf{C}$-linear $\frac{\partial A \circ R}{\partial z_j} = A \circ \frac{\partial R}{\partial z_j}$, as it follows immediately from the definition of $\frac{\partial}{\partial z_j}$ and of the commutation of A and J_{st}. So, if $Q(p) = 0$, one has $\frac{\partial J}{\partial z_j}(p) = -2J_{st} \circ \frac{\partial \overline{Q}}{\partial z_j}(p)$ and similarly for the $\overline{z}_j$ derivatives. When $Q(p) \neq 0$, there is no simple relation between the z_j derivatives of J and of $\overline{Q}$ (see 2.4.3).

2.3) *More on the operator Q and J-holomorphic discs. The Diederich-Sukhov normalization*

Differentiation of $(**)$, with respect to $\overline{\zeta}$, gives the following formula for the Laplacian of u:

$$\frac{1}{4}\overline{\Delta u} \ = \ \frac{\partial^2 \overline{u}}{\partial \zeta \partial \overline{\zeta}} = \Big[Q_z(u).\frac{\partial u}{\partial \overline{\zeta}}\Big]\Big(\frac{\partial u}{\partial \zeta}\Big) + \Big[Q_{\overline{z}}(u).\frac{\partial \overline{u}}{\partial \overline{\zeta}}\Big]\Big(\frac{\partial u}{\partial \zeta}\Big) + Q(u)\frac{\partial^2 u}{\partial \zeta \partial \overline{\zeta}} \ .$$

Proposition 2

Let $p \in \mathbf{C}^n$. Assume that J is a $\mathcal{C}^\alpha$ almost complex structure defined near p, $\alpha > 1$, and that $J(p) = J_{st}$. Then every J-holomorphic disc u with $u(0) = p$ satisfies $\frac{\partial u}{\partial \overline{\zeta}}(0) = 0$, and the following are equivalent:

(i) *Every J-holomorphic disc u with $u(0) = p$, satisfies $\frac{\partial^2 u}{\partial\zeta\partial\overline{\zeta}}(0) = 0$,*

(ii) *$Q_{\overline{z}}(p) = 0$.*

Proof. Since $\alpha > 1$, the J holomorphic discs are of class $\mathcal{C}^2$. Since $J(p) = J_{st}$, $Q(p) = 0$. If $u(0) = p$, $\frac{\partial u}{\partial \overline{\zeta}}(0) = \overline{\frac{\partial \overline{u}}{\partial \zeta}}(0) = 0$ by $(**)$. The formula above for the Laplacian of u reduces to $\frac{\partial^2 \overline{u}}{\partial\zeta\partial\overline{\zeta}}(0) = [Q_{\overline{z}}(p).\overline{T}](T) = 0$, for $T = \frac{\partial u}{\partial \zeta}(0)$ $\quad (= \frac{\partial u}{\partial x}(0)$ since $J(p) = J_{st})$. Clearly, (ii) implies (i). Next, if θ is a $\mathbf{C}$-bilinear form on $\mathbf{C}^n$, no symmetry assumed, it is equivalent to have $\theta(S, T) = 0$ for all S and $T \in \mathbf{C}^n$ and to have $\theta(\overline{T}, T) = 0$ for all $T \in \mathbf{C}^n$. This elementary remark (and the fact that for any $T \in \mathbf{C}^n$ there exists a J-holomorphic disc u with $u(0) = p$ and $\frac{\partial u}{\partial x}(0) = \lambda T$, for some $\lambda > 0$,) allows one to conclude that (i) implies (ii). Proposition 2 is in the same spirit as [10, Lemma 2.5].

Theorem 1 (Diederich-Sukhov) ([1, Lemma 3.2])

Let J be an almost complex structure of class $\mathcal{C}^\alpha$, $\alpha > 1$, defined near p in $\mathbf{C}^n$. Then, one can make a quadratic change of coordinates, so that in the new coordinates: $p = 0$, $J(0) = J_{st}$ and $Q_{\overline{z}}(0) = 0$.

Our proof is not the proof given by Diederich and Sukhov, it is instead an immediate application of Proposition 2. Of course we can assume that $p = 0$ and that already $J(0) = J_{st}$. Then, take new variables Z defined by

$$Z = z - \overline{[Q_{\overline{z}}(0).\overline{z}](z)} \qquad \text{i.e. more conveniently for us}$$
$$\overline{Z} = \overline{z} + [Q_{\overline{z}}(0).\overline{z}](z) \ .$$

If u is a J-holomorphic disc with $u(0) = 0$, we have

$$\frac{\partial u}{\partial \overline{\zeta}}(0) = 0 \ , \quad \text{and} \quad \frac{\partial^2 \overline{u}}{\partial\zeta\partial\overline{\zeta}} = \Big[Q_{\overline{z}}(0).\overline{\frac{\partial u}{\partial \zeta}}\Big]\Big(\frac{\partial u}{\partial \zeta}\Big) \ .$$

To u corresponds in the new coordinates a disc U with

$$\overline{U} = \overline{u} - [Q_{\overline{z}}(0).\overline{u}](u) \ ,$$

which, due to $u(0) = 0$ and $\frac{\partial u}{\partial \overline{\zeta}}(0) = 0$, is immediately seen to satisfy $\frac{\partial U}{\partial \overline{\zeta}}(0) = 0$, and additionally $\frac{\partial^2 \overline{U}}{\partial\zeta\partial\overline{\zeta}}(0) = 0$, equivalently $\frac{\partial^2 U}{\partial\zeta\partial\overline{\zeta}}(0) = 0$. □

Comment. Although the theorem is elementary, it may be worth making a non elementary comment. It is clear that the condition $Q_{\overline{z}}(p) = 0$ would be satisfied if all the standard complex lines through p were J-holomorphic. A non-elementary change of variables has been made by Duval [2] for reaching that situation, for blow up. Unfortunately smoothness of J is not preserved. The problem here was much simpler, at the level of second order Taylor expansion.

2.4) $J_{\overline{z}} = 0$, $Q_z = 0$ *and vanishing of the Nijenhuis tensor*

2.4.1. Preliminaries

The non vanishing of the Nijenhuis tensor is the obstruction for an almost complex structure to be a complex structure. The easiest way to get this tensor is by working with the complexified tangent bundle to $\mathbf{C}^n = \mathbf{R}^{2n}$ (see Proposition 4 below). All vector fields under consideration in this section will be at least of class $\mathcal{C}^1$ (except their Lie brackets that will be at least continuous) and the almost complex structure J will also be at least of class $\mathcal{C}^1$. A complexified vector field θ is said to be of type $(0,1)$ at p if $\theta(p) = X + iJ(p)X)$, for some (real) tangent vector X, the vector field is said to be of type $(0,1)$ if it is of type $(0,1)$ at each point (we shall use notation such as $\overline{L}$ for such vector fields). An almost complex structure is a complex structure if and only if there exist local coordinates $Z = (Z_1, \dots, Z_n)$ such that $\overline{L}(Z_j) = 0$ for any $(0,1)$ vector field $\overline{L}$. If the structure is smooth enough, for this to happen, it is necessary and sufficient that the Lie Bracket of any two vector fields of type $(0,1)$ be of type $(0,1)$. Necessity is obvious. Sufficiency follows immediately from the Frobenius theorem in case of real analytic data (see e.g. [7, p.125-6]), and in the smooth case this is the Newlander-Nirenberg Theorem.

Recall the definition of the Nijenhuis tensor, that does not require any complexification of the tangent bundle: Let X and Y be two (real) vector fields defined near $p \in \mathbf{C}^n$, set

$$N(X,Y) = [X,Y] + J[JX,Y] + J[X,JY] - [JX,JY] \ .$$

Note: The value of $N(X,Y)$ at a point depends only on the value of X and Y at that point (the Nijenhuis tensor is indeed a tensor). Equivalently (since $N(Y,X) = -N(X,Y)$), $N(X,Y)$ vanishes at any point where Y vanishes. When working with the complexified tangent bundle as in Proposition 4 below, this fact essentially reduces to the fact that if L and M are sections of a sub-bundle of the (real or complexified) tangent bundle, then their Lie Bracket lies in that sub-bundle at each point where either L or M vanishes. Here we do a direct checking that $N(X,Y) = 0$ at any point where $Y = 0$. Note that $J\big((JX)Y - Y(JX)\big)$ makes sense, since by cancellation of second order terms $[JX,Y]$ is a vector field, but that neither $J\big((JX)Y\big)$ nor $J\big(X(JY)\big)$ make sense. Instead, there is a basic fact, stated in a more general setting, that one can use (a fact that is essentially not more than $(uv)'(0) = u(0)v'(0)$, if $v(0) = 0$):

Let X and Y be now vector fields defined near 0 on $\mathbf{R}^d$, with variable denoted by $t = (t_1, \dots, t_d)$.

$$X(t) = \sum_j a_j(t) \frac{\partial}{\partial t_j} \ , \ Y(t) = \sum_j b_j(t) \frac{\partial}{\partial t_j},$$

where a_j and b_j are $\mathcal{C}^1$ functions. And let C be a $(d \times d)$ matrix valued function, $C = (C_{j,k})_{1 \leq j,k \leq d}$. So, C acts on vector fields, e.g. $CX = \sum_j (\sum_k c_{j,k} a_k) \frac{\partial}{\partial t_j}$. We claim that if $Y(0) = 0$, then $[X, CY](0) = C[X,Y](0)$. It is enough to check the claim with $X = \frac{\partial}{\partial t_1}$, $Y = b \frac{\partial}{\partial t_r}$ for some r and some function b with $b(0) = 0$. In that case,

$[X,Y](0) = \frac{\partial b}{\partial t_1}\frac{\partial}{\partial t_r}$. So, using $b(0)=0$ for the second equality:

$$C[X,Y](0) = \sum_j c_{j,r}\frac{\partial b}{\partial t_1}(0)\frac{\partial}{\partial t_j} = \sum_j \frac{\partial c_{j,r} b}{\partial t_1}(0)\frac{\partial}{\partial t_j} = [X, CY](0)\ .$$

We are now ready for proving that $N(X,Y)(0) = 0$ if $Y(0) = 0$ (going back to the previous setting of $\mathbf{R}^{2n}$). If $Y(0) = 0$, then using the above, we get: $J[X,JY](0) = [X, J^2Y](0) = -[X,Y](0)$, and $J[JX,Y](0) = [JX,JY](0)$. So $N(X,Y)(0) = 0$.

Proposition 3

The following are equivalent:

(a) $J(p) = J_{st}$.
(b) $\overline{L}(z_j)\ (p) = 0$ *for all* $j \in \{1,\dots,n\}$ *and every* $(0,1)$ *vector field* $\overline{L}$.

Then, for a vector field θ *to be of type* $(0,1)$ *at* p*, it is necessary and sufficient that* $\theta(z_j)\ (p) = 0$.

These are simple algebraic facts.

Proposition 4

For p *fixed, the following are equivalent:*

(i) $N(X,Y)(p) = 0$ *, for all (real) tangent vectors* X *and* Y.
(ii) *For every* $(0,1)$ *vector field* $\overline{L}$ *and* $\overline{M}$*, their Lie Bracket* $[\overline{L},\overline{M}]$ $(= \overline{LM} - \overline{ML})$ *is of type* $(0,1)$ *at the point* p.
(iii) *There exist a (quadratic) local change of coordinates* $z \to Z(z) = (Z_1,\dots,Z_n)$ *near* p *such that* $Z(p) = 0$*, and* $\overline{L}(Z_j)(z) = o(|z-p|)$ *(as* $z \to p$*), for all* $(0,1)$ *vector fields* $\overline{L}$.

Note: The last condition can be written as $\overline{\partial}_J Z_j = o(|z|)$ where for a function ψ, $\overline{\partial}_J\psi$ is the 1-form defined by

$$\overline{\partial}_J\psi(X) = \frac{1}{2}\big(d\psi(X) + i d\psi(JX)\big)\ ,$$

for any tangent vector X. If $J = J_{st}$ we simply write $\overline{\partial}$, and this is in agreement with the usual definition $\overline{\partial}\psi = \sum_j \frac{\partial\psi}{\partial\overline{z}_j}d\overline{z}_j$.

Proof. (ii) is equivalent to (i). Indeed $[X+iJX, Y+iJY] = [X,Y] - [JX,JY] + i([X,JY] + [JX,Y])$. So, $[X+iJX, Y+iJY]$ is of type $(0,1)$ at p if and only if $[X,JY](p) + [JX,Y](p) = J(p)\big([X,Y](p) - [JX,JY](p)\big)$. Multiply both sides by $J(p)$ and use $J^2 = -\mathbf{1}$ to get the desired conclusion.

Assume that there is a $\mathcal{C}^1$ change of coordinates as in (iii). then $\overline{L}(Z_j)$ and $\overline{M}(Z_j) = o(|z-p|)$, for all $(0,1)$ vector fields $\overline{L}$ and $\overline{M}$. Consequently $\big(\overline{LM}(Z_j)\big)(p) = 0$ and $\big(\overline{ML}(Z_j)\big)(p) = 0$. Hence $[\overline{L},\overline{M}](p)(Z_j) = 0$. (ii) then follows from Proposition 3.

Finally we check that (ii) implies (iii). We can assume that $p = 0$ and that $J(0) = J_{st}$. Then by simple linear algebra, there is a basis of $(0,1)$ vector fields $\overline{L}_j$ such that

$$\overline{L}_j = \frac{\partial}{\partial\overline{z}_j} + \sum_q \alpha_{j,q}\frac{\partial}{\partial z_q}\ ,$$

where the $\alpha_{j,k}$ are functions that vanish at 0. Then (ii) has the following simple form: $[\overline{L}_j, \overline{L}_k](0) = 0$. That gives us

$$\frac{\partial \alpha_{j,q}}{\partial \overline{z}_k}(0) = \frac{\partial \alpha_{k,q}}{\partial \overline{z}_j}(0) \ .$$

We now look for a change of variables $z \mapsto Z(z)$ such that $\overline{L}_j(Z_r) = o(|z|)$. We take a simple quadratic change of variables:

$$Z_r(z) = z_r + \sum_{k,l} a^r_{k,l} z_k \overline{z}_l + \sum_{k,l} b^r_{k,l} \overline{z}_k \overline{z}_l \ .$$

We then have:

$$\overline{L}_j(Z_r) = \sum_k a^r_{k,j} z_k + \sum_k (b^r_{k,j} + b^r_{j,k}) \overline{z}_k + \alpha_{j,r} + o(|z|) \ .$$

We then take the constants $a^r_{k,j} = -\frac{\partial \alpha_{j,r}}{\partial z_k}(0)$ and $b^r_{k,j}$ such that $b^r_{k,j} + b^r_{j,k} = -\frac{\partial \alpha j,r}{\partial \overline{z}_k}(0)$. This last choice is possible (by symmetry) if and only if one has the compatibility condition $\frac{\partial \alpha_{j,r}}{\partial \overline{z}_k}(0) = \frac{\partial \alpha_{k,r}}{\partial \overline{z}_j}(0)$, which is indeed satisfied.

2.4.2. End of the discussion of the Diederich-Sukhov normalization

We start with an easy remark on the case when J matches with J_{st} to order 1.

Proposition 5 (With hypotheses as in 2.4.1)

The following are equivalent:

(i) $N(p) = 0$.

(ii) *In appropriate coordinates* $p = 0$, $J(0) = J_{st}$, $\nabla J\ (0) = 0$.

Proof. (ii) implies (i) trivially since evaluating the Nijenhuis tensor N requires only one derivative of J. If (i) is satisfied, in coordinates given by Proposition 4, (ii) is satisfied. Recall that $\nabla J = 0$ is equivalent to $\nabla Q = 0$ at a point where $J = J_{st}$.

Next we want to end the discussion of the Diederich and Sukhov normalization by writing down the proof of a result that they stated, and that illustrates how different are the requirements $Q_z = 0$ and $Q_{\overline{z}} = 0$. The first one can be achieved at a point where $J = J_{st}$ only if the Nijenhuis tensor vanishes at that point.

Proposition 6

If $J(p) = J_{st}$ *and* $J_{\overline{z}}(p) = 0$ *(equivalently* $Q_z(p) = 0$*), then* $N(p) = 0$.

For the converse see Proposition 5. However note that the conditions $J(p) = J_{st}$ and $N(p) = 0$ do not imply that, in arbitrary coordinates, $J_{\overline{z}}(p) = 0$, even if one additionally assumes that $J_z(p) = 0$. Indeed even in complex dimension 1 (so $N = 0$ automatically), consider the structure given by $Q(z) = \overline{z}$. It is instructive to discuss this elementary example, that is so simple to describe in terms of Q and not so immediate in terms of J. This is done in 2.4.3 at the end of Section 2. Proposition 6 follows from Proposition 4 and the following Lemma:

Lemma

If $J(0) = J_{st}$, and $J_{\overline{z}}(0) = 0$, then there exist local coordinates $Z = Z(z)$, with $Z(0) = 0$, such that $\overline{\partial}_J Z = o(|z|)$.

Proof. Since $J_{\overline{z}}(0) = 0$, $J(z) = J_{st} + \sum_k z_k \epsilon_k + o(|z|)$, where $\epsilon_k = \frac{\partial J}{\partial z_k}(0)$. For any function ψ and any tangent vector $X = (X_1, \dots X_n)$ $(\in \mathbf{C}^n)$,

$$\overline{\partial}_J \psi(X) = \overline{\partial}\psi(X) + i\ d\psi\Big(\sum z_k \epsilon_k(X)\Big) + o(|z|)\ .$$

Taking $\psi = z_j$, whose differential is $\mathbf{C}$-linear, one gets

$$\overline{\partial}_J z_j\ (X) = \sum_k z_k\ dz_j(\epsilon_k(X)) + o(|z|)\ .$$

As pointed out earlier, in Note (i) at the end of 2.1, as an immediate consequence of $J^2 = -\mathbf{1}$ and $J(0) = 0$, each ϵ_k is a conjugate linear map from $\mathbf{C}^n$ to $\mathbf{C}^n$. So, by $\mathbf{C}$-linearity of dz_j, $dz_j \circ \epsilon_k$ is a conjugate linear map from $\mathbf{C}^n$ to $\mathbf{C}$, i.e. one can write $dz_j \epsilon_k(X) = \sum_l \mu^l_{j,k} \overline{X}_l$ (with no X_l terms). Set $Z_j = z_j + \sum_k z_k A_{j,k}(z)$, where the functions $A_{j,k}$ are linear functions (vanishing at 0) to be found. Since at the origin $\overline{\partial}_J$ coincides with $\overline{\partial}$, and since terms of order > 1 are discarded, the functions $A_{j,k}$ simply have to satisfy: $\overline{\partial} A_{j,k}(X) = -dz_j \epsilon_k(X)$, which is possible since the right hand side is conjugate linear in X. This ends the proof of the Lemma and Proposition 6.

Remark 1 Trying to do the proof of the Lemma with $J_z = 0$ instead of $J_{\overline{z}} = 0$ in the hypothesis leads to study the case $J = J_{st} + \sum_k \overline{z}_k \eta_k$. Instead of $z_k A_{j,k}$, whose $\overline{\partial}$ was simply $z_k \overline{\partial} A_{j,k}$, we could start by considering expressions such as $\overline{z}_k A_{j,k}$, whose $\overline{\partial}$ leads to immediate difficulties. Compatibility conditions, as in the proof of (ii) implies (iii) in Proposition 4, arise.

2.4.3. Here we discuss the elementary example mentioned after the statement of Proposition 6

This is to illustrate how the theoretically trivial switch from the $\mathbf{C}$-linear operator Q to the operator J which is only $\mathbf{R}$-linear, is in fact not so pleasant computationally, and to see what actual computations may be. Here is a sketch of the un-enlightening computations.

We start with the almost complex structure defined near 0 on $\mathbf{C}$ (with variable $z_1 = x_1 + iy_1$) by $Q(z_1) = \overline{z}_1$. So the equation for J-holomorphic discs ($u = u(\zeta)$, with $\zeta = x + iy$ in order to keep our previous notations), is

$$\frac{\partial \overline{u}}{\partial \zeta} = \overline{u}\frac{\partial u}{\partial \zeta} \qquad \text{i.e.} \quad \frac{\partial u}{\partial \overline{\zeta}} = u\ \overline{\frac{\partial u}{\partial \zeta}}\ .$$

A way to get J is by separating real parts and imaginary parts in the equations above and to write that

$$\begin{pmatrix} \mathrm{Re}\ \frac{\partial u}{\partial y} \\ \mathrm{Im}\ \frac{\partial u}{\partial y} \end{pmatrix} \ = \ J(u) \begin{pmatrix} \mathrm{Re}\ \frac{\partial u}{\partial x} \\ \mathrm{Im}\ \frac{\partial u}{\partial x} \end{pmatrix}.$$

Another possibility is to use the formula $J = J_{st} \circ [\mathbf{1} - \overline{Q}][\mathbf{1} + \overline{Q}]^{-1}$. We took Q to be the $\mathbf{C}$ linear map from $\mathbf{C}$ to $\mathbf{C}$ defined by: $Q(t) = \overline{z}_1 t$. So $\overline{Q}$ is the conjugate linear map: $t \mapsto z_1\bar{t}$, and thus, in $\mathbf{R}^2$ notations $\overline{Q}$ is represented by the matrix $\begin{pmatrix} x_1 & y_1 \\ y_1 & -x_1 \end{pmatrix}$. At any rate the conclusion is that the matrix J defining the almost complex structure is given by

$$J(z_1) = \frac{1}{1-|z_1|^2}\begin{pmatrix} 2y_1 & -1-|z_1|^2-2x_1 \\ 1+|z_1|^2-2x_1 & -2y_1 \end{pmatrix}.$$

Of course one has $J^2 = -\mathbf{1}$. Our choice of Q was such that $\frac{\partial}{\partial z_1}Q = 0$ (everywhere). This has no clear interpretation in terms of J, except at the points where $J = J_{st}$, i.e. for $z_1 = 0$, for which computations are immediate.

$$\frac{\partial}{\partial y_1}J\ (0) = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -2 \\ -2 & 0 \end{pmatrix} = J_{st} \circ \frac{\partial}{\partial x_1}J\ (0)$$

which is the definition of $\frac{\partial J}{\partial \bar{z}_1}(0) = 0$.

But at arbitrary point z_1, $\frac{\partial J}{\partial \bar{z}_1}$ does not vanish. It has already been pointed out in **2.2** that the formula $J = J_{st}[\mathbf{1} - \overline{Q}][\mathbf{1} + \overline{Q}]^{-1}$ $(= J_{st} \circ [\mathbf{1} + 2\sum_{k\geq 1}(-1)^k\overline{Q}^k]$, if Q has operator norm < 1) does not yield any simple relation betwen the derivatives of $\overline{Q}$ and J at points where $Q \neq 0$. The following simple fact is enough explanation. We have $\overline{Q}(t) = z_1\bar{t}$, so $\overline{Q}^2(t) = z_1\bar{z}_1 t$. Hence $\overline{Q}_{\bar{z}_1} = 0$ while $(\overline{Q}^2)_{\bar{z}_1} \neq 0$.

3. The Sukhov-Tumanov normalization

See [10, 11, Sections 2.2]. A proof is written in detail in [10] for complex dimension 2. The case of arbitrary dimension is not completely clearly stated since the statement of [11, Lemma 2.2] is preceded by a comment on the fact that (in the present notations) one can assume that Q is close to 0 in some $\mathcal{C}^k$ norm. That restriction is however not needed in the Lemma.

Let u be a map from $\overline{\mathbf{D}}$, the closed unit disc in $\mathbf{C}$ into an almost complex manifold, that is a J-holomorphic embedding of class $\mathcal{C}^\alpha$. Then, one can find complex coordinates, of class $\mathcal{C}^\alpha$ on a neighborhood of the disc such that in these coordinates $u(\overline{\mathbf{D}})$ is the closed unit disc on the z_1 axis, and the almost complex structure coincides with the standard one along that disc. First, one applies the implicit function theorem and the triviality of the normal bundle, and then simple linear algebra to get $J = J_{st}$. Then if one accepts to change the almost complex structure in the region $|z_1| \geq 1$, one can assume that $J = J_{st}$ along the whole z_1 axis, just in order to simplify the statements.

Without any further discussion of these facts, we now focus on the essential and that will be stated in Theorem 2.

We shall denote by e_1 the first vector in the standard basis of $\mathbf{C}^n$, $e_1 = (1, 0, \ldots, 0)$; $\overline{\mathbf{D}}_r$ will denote the closed disc of radius r in $\mathbf{C}$ and for $r = 1$ we shall drop the index.

Before stating Theorem 2, some words about smoothness: In Theorem 2, $J = J_{st}$ along the z_1 axis, so, along the z_1 axis $J_z = -2J_{st} \circ \overline{Q}_z = -2J_{st} \circ \overline{Q_{\bar{z}}}$. Therefore the smoothness hypotheses given can equivalently be given in terms of J_z or in terms of $Q_{\overline{z}}$. We felt that it is worth making a distinction between the smoothness of the almost complex structure J (equivalently of Q) and the smoothness of J_z (equivalently $Q_{\overline{z}}$). Of course, if J is of class $\mathcal{C}^\alpha$, J_z is at least of class $\mathcal{C}^{\alpha-1}$, but it can be smoother (and in case $J_z = 0$ on the z_1 axis, no change of variable is required!). In [10, Remark 2], Sukhov and Tumanov mention the question of smoothness, without making clear the assumption of smoothness of J, rather focusing on the smoothness of the disc, which, going a step ahead, we immediately took to be the unit disc in the z_1 axis. Our effort has been on the clarity of the proof, and breaking the proof in two steps is likely the origin of a loss (since in case $\beta = \alpha - 1$ Sukhov and Tumanov seem to claim a better result). Recall that if J is an almost complex structure of class $\mathcal{C}^\alpha$ and χ is a $\mathcal{C}^\beta$ change of variables, the resulting almost complex structure is of class $\mathcal{C}^\gamma$ with $\gamma = \min(\alpha, \beta - 1)$.

Theorem 2

Let J be an almost complex structure of class $\mathcal{C}^\alpha$, $\alpha > 1$, defined on a neighborhood of $\overline{\mathbf{D}} \times \{0\}$ in $\mathbf{C}^n = \mathbf{C} \times \mathbf{C}^{n-1}$. Assume that $J = J_{st}$ along the z_1 axis (that is thus a J-holomorphic curve), and that the map $z_1 \mapsto J_z(z_1, 0, \ldots, 0)$ is of class $\mathcal{C}^\beta$, with $\beta > 3$, β not an integer.

Then, there is $r > 0$, and a $\mathcal{C}^{\beta-1}$ diffeomorphism $z \mapsto Z(z)$ from $\overline{\mathbf{D}} \times \overline{\mathbf{D}}_r^{n-1}$ into $\mathbf{C}^n$, with $Z(z_1, 0, \ldots, 0) = (z_1, 0, \ldots, 0)$, such that in the coordinates provided by Z, $J = J_{st}$ and $J_Z = 0$ (equivalently $Q_{\overline{Z}} = 0$) along $\overline{\mathbf{D}} \times \{0\}$.

Proof. The proof can be clearly broken in two steps, that we now quickly describe.

Step 1: Let $e_1 = (1, 0, \ldots, 0)$ be the first vector in the standard basis of $\mathbf{C}^n$. We want to get $J_z = 0$ along the z_1 axis, equivalently $Q_{\overline{z}} = 0$. The first step consists in getting only $Q_{\overline{z}}(e_1) = 0$ (i.e. $\frac{\partial}{\partial \overline{z}_k}(Q(z)e_1) = 0$, $k = 1, \ldots, n$) along the z_1 axis, more precisely on a a neighborhood of the closed unit disc on the z_1 axis. That is to say that we do the job only for the first column of the $(n \times n)$ complex matrix Q. This will be done by a $\mathcal{C}^\beta$ change of variables.

Step 2: That is elementary, is the end of the proof, assuming that we already have $Q_{\overline{z}}(e_1) = 0$ along the z_1 axis.

Note that along the z_1 axis, since $Q = 0$ we already have $\frac{\partial Q}{\partial \overline{z}_1} = 0$.

Notation $=^o$. In the proof, many equations, for J-holomorphic discs u, will be written that are valid only if $u(0)$ belongs to the unit disc in the z_1 axis. We wish to emphasize that and we shall use $=^o$ instead of $=$, for equalities that hold at $\zeta = 0$ under the hypothesis that $u_2(0) = \ldots, u_n(0) = 0$, e.g. we have $\frac{\partial(u_1 u_2)}{\partial \zeta} =^o u_1 \frac{\partial u_2}{\partial \zeta}$. Another example, much used below, is $[Q_{\overline{z}} . \overline{\frac{\partial u}{\partial \zeta}}] =^o \sum_{j \geq 2} \overline{\frac{\partial u_j}{\partial \zeta}} \frac{\partial Q}{\partial \overline{z}_j}(u)$, where the term with $j = 1$ need not to be written in the right hand side, since $\frac{\partial Q}{\partial \overline{z}_1} = 0$ along the z_1 axis.

Since Step 2 is a mere repetition of the proof of Theorem 1 (to be read first), with z_1 as a parameter, we begin with Step 2.

Proof of Step 2. Our goal being to have $\overline{\partial} Q = 0$ along the z_1 axis, we assume that we already have along the z_1 axis $\overline{\partial}(Q(z)e_1) = 0$. We shall reach this situation with

Step 1, using a $\mathcal{C}^\beta$ change of variables with vanishing $\overline{\partial}$ along the z_1 axis. By immediate application of chain rule, one see that in these new coordinates $z_1 \mapsto J_{\overline{z}}(z_1, 0, \dots, 0)$ is of class $\mathcal{C}^{\beta-1}$, and J is of class $\mathcal{C}^{\min\,(\alpha,\beta-1)}$.

Recall that Q is given by an $(n \times n)$ complex matrix, that then $Q_{\overline{z}}$ can be simply understood by coefficient-wise differentiation of the matrix, and that Q comes in the equations satisfied by J-holomorphic discs.

$$\frac{\partial \overline{u}}{\partial \zeta} = Q(u) \frac{\partial u}{\partial \zeta} \ ,$$

and

$$\frac{\partial^2 \overline{u}}{\partial \zeta \partial \overline{\zeta}} = \Big[Q_{\overline{z}} . \overline{\frac{\partial u}{\partial \zeta}} \Big] \Big(\frac{\partial u}{\partial \zeta} \Big) \ . \tag{E}$$

Since $Q = 0$ on the z_1 axis, $\frac{\partial u}{\partial \overline{\zeta}}(0) =^o 0$. We have already pointed out that $\frac{\partial Q}{\partial \overline{z}_1} = 0$ along the z_1 axis. Hence

$$Q_{\overline{z}}(z_1, 0, \dots, 0).(t_1, t_2, \dots, t_n) = \sum_j t_j \frac{\partial Q}{\partial \overline{z}_j} = Q_{\overline{z}}(z_1, 0, \dots, 0).(0, t_2, \dots, t_n) \ .$$

Now we shall use the hypothesis that, for $j = 1, \dots, n$,

$$\frac{\partial}{\partial \overline{z}_j} (Q(z) e_1) = 0 \text{ along the } z_1 \text{ axis} \ ,$$

i.e. the first column of Q is already in the kernel of $\overline{\partial}$. Then (E) gives us

$$\frac{\partial^2 \overline{u}}{\partial \zeta \partial \overline{\zeta}}(0) =^o \Big[Q_{\overline{z}}(z_1, 0, \dots, 0). \Big(0, \overline{\frac{\partial u_2}{\partial \zeta}}(0), \dots, \overline{\frac{\partial u_n}{\partial \zeta}(0)} \Big) \Big] \Big(0, \frac{\partial u_2}{\partial \zeta}(0), \dots, \frac{\partial u_n}{\partial \zeta}(0) \Big) \ . \tag{E'}$$

(In matrix multiplication, remember to treat $(0, \frac{\partial u_2}{\partial \zeta}(0), \dots, \frac{\partial u_n}{\partial \zeta}(0))$ as a column vector.)

Treating somewhat z_1 as a parameter, we can now do the same simple $\mathcal{C}^{\beta-1}$ change of variables as in the proof of Theorem 1 (easier to write for the conjugates), setting:

$$(\overline{Z}_1, \dots, \overline{Z}_n) = (\overline{z}_1, \overline{z}_2, \dots, \overline{z}_n) + \big[Q_{\overline{z}}(z_1, 0, \dots 0).(0, \overline{z}_2, \dots, \overline{z}_n) \big] (0, z_2, \dots, z_n) \ .$$

This last change of variables leads to an almost complex structure of class $\mathcal{C}^\gamma$ with $\gamma = \min(\alpha, \beta - 2) > 1$, since $\alpha > 1$, $\beta > 3$, so one can apply Proposition 2. To any J-holomorphic disc $u = (u_1, \dots, u_n)$, with $u(0) = (z_1, 0, \dots, 0)$, corresponds a disc $U = (U_1, \dots, U_n)$ with

$$\overline{U} = \overline{u} + \big[Q_{\overline{z}}(z_1, 0, \dots, 0).(0, \overline{u}_2, \dots, \overline{u}_n) \big] (0, u_2, \dots, u_n) \ .$$

Exactly as in the proof of Theorem 1, U obviously satisfies $U(0) = (z_1, 0, \dots, 0)$, $\frac{\partial U}{\partial \overline{\zeta}}(0) = 0$ and finally, following from (E'),

$$\frac{\partial^2 U}{\partial \zeta \partial \overline{\zeta}}(0) = \overline{\frac{\partial^2 \overline{U}}{\partial \zeta \partial \overline{\zeta}}}(0) = 0 \ .$$

By Proposition 2, this shows that in the new coordinates along $\mathbf{C} \times \{0\}$, $J = J_{st}$, and $Q_{\overline{z}} = 0$.

Step 1, case of dimension 2. For convenience of the reader we start with the case of dimension 2. This case is not really different but computations may be easier to follow, and full detail is easier to provide.

Write $u = (u_1, u_2)$. Recall that $\frac{\partial Q}{\partial \overline{z}_1}(z_1, 0) = 0$. So (E) reduces to:

$$\frac{\partial^2 \overline{u}_1}{\partial \zeta \partial \overline{\zeta}}(0) =^o \overline{\frac{\partial u_2}{\partial \zeta}} \Big(a_{1,1}(u_1) \frac{\partial u_1}{\partial \zeta} + a_{1,2}(u_1) \frac{\partial u_2}{\partial \zeta} \Big)$$

$$\frac{\partial^2 \overline{u}_2}{\partial \zeta \partial \overline{\zeta}}(0) =^o \overline{\frac{\partial u_2}{\partial \zeta}} \Big(a_{2,1}(u_1) \frac{\partial u_1}{\partial \zeta} + a_{2,2}(u_1) \frac{\partial u_2}{\partial \zeta} \Big)$$

where the right hand side is evaluated at $\zeta = 0$, and with

$$a_{p,k}(z_1) = \frac{\partial q_{p,k}}{\partial \overline{z}_2}(z_1, 0) \ .$$

The goal is to have $a_{1,1} = a_{2,1} = 0$, and keep $Q = 0$ on the z_1 axis For this, one makes a change of variables, leaving the z_1 axis invariant and preserving J_{st} along the z_1 axis, given by:

$$Z_1 = z_1 + z_2 f(z_1) \qquad , \qquad Z_2 = z_2 e^{g(z_1)} \ ,$$

where f and g are functions to chose appropriately. To any (germ of) J-holomorphic disc $u = (u_1, u_2)$ corresponds a J-holomorphic disc $U = (U_1, U_2)$ with

$$U_1 = u_1 + u_2 f(u_1) \ , \ U_2 = u_2 e^{g(u_1)} \ . \tag{F}$$

We have $\frac{\partial u_j}{\partial \overline{\zeta}} =^o 0$ and $\frac{\partial U_j}{\partial \overline{\zeta}} =^o 0$, and we get

$$\frac{\partial U_1}{\partial \zeta} =^o \frac{\partial u_1}{\partial \zeta} + \frac{\partial u_2}{\partial \zeta} f(u_1)$$
$$\frac{\partial U_2}{\partial \zeta} =^o \frac{\partial u_2}{\partial \zeta} e^{g(u_1)} \ .$$

Therefore

$$\frac{\partial u_1}{\partial \zeta} =^o \frac{\partial U_1}{\partial \zeta} - \frac{\partial U_2}{\partial \zeta} f(u_1) e^{-g(u_1)}$$
$$\frac{\partial u_2}{\partial \zeta} =^o \frac{\partial U_2}{\partial \zeta} e^{-g(u_1)} \ .$$

Taking the Laplacian of each side in the (conjugate of the) equation (F) leads to

$$\frac{\partial^2 \overline{U}_1}{\partial \zeta \partial \overline{\zeta}} =^o \frac{\partial^2 \overline{u}_1}{\partial \zeta \partial \overline{\zeta}} + \frac{\partial^2 \overline{u}_2}{\partial \zeta \partial \overline{\zeta}} \overline{f}(u_1) + \overline{\frac{\partial u_2}{\partial \zeta}} \frac{\partial \overline{f}}{\partial z_1}(u_1) \frac{\partial u_1}{\partial \zeta} \ .$$

So,

$$\frac{\partial^2\overline{U}_1}{\partial\zeta\partial\overline{\zeta}} =^o \overline{\frac{\partial u_2}{\partial\zeta}}\Big(a_{1,1}(u_1)\frac{\partial u_1}{\partial\zeta} + a_{1,2}(u_1)\frac{\partial u_2}{\partial\zeta}\Big) + \overline{\frac{\partial u_2}{\partial\zeta}}\Big(a_{2,1}(u_1)\frac{\partial u_1}{\partial\zeta} + a_{2,2}(u_1)\frac{\partial u_2}{\partial\zeta}\Big)\overline{f}(u_1) + \overline{\frac{\partial u_2}{\partial\zeta}}\frac{\partial\overline{f}}{\partial z_1}(u_1)\frac{\partial u_1}{\partial\zeta}$$

and
$$\frac{\partial^2\overline{U}_2}{\partial\zeta\partial\overline{\zeta}} =^o \overline{\frac{\partial u_2}{\partial\zeta}}\Big(a_{2,1}(u_1)\frac{\partial u_1}{\partial\zeta} + a_{2,2}(u_1)\frac{\partial u_2}{\partial\zeta}\Big)e^{\overline{g}(u_1)} + \overline{\frac{\partial u_2}{\partial\zeta}}\frac{\partial\overline{g}}{\partial z_1}e^{\overline{g}(u_1)}\frac{\partial u_1}{\partial\zeta} \ .$$

Then, one would have to express the $\frac{\partial u_j}{\partial\zeta}$'s in terms of the $\frac{\partial U_j}{\partial\zeta}$'s. Doing this work is however not needed. Simply observe that having no term with a $\frac{\partial U_1}{\partial\zeta}$ after this substitution is equivalent to have cancellation of the $\frac{\partial u_1}{\partial\zeta}$ factors in the right hand side of the above equations. So, we are led to the conditions that f and g must satisfy (all functions being functions of z_1):

$$a_{1,1} + a_{2,1}\overline{f} + \frac{\partial\overline{f}}{\partial z_1} = 0$$

$$a_{2,1} + \frac{\partial\overline{g}}{\partial z_1} = 0 \ .$$

These equations are easily solved. The second one is just the standard $\overline{\partial}$ problem. By setting $f = e^{\alpha}F$ with $\frac{\partial\overline{\alpha}}{\partial z_1} = -a_{2.1}$, the first equation reduces to $\frac{\partial\overline{F}}{\partial z_1} + e^{-\overline{\alpha}}a_{1,1} = 0$.

The coefficients in the equations are of class $\mathcal{C}^{\beta-1}$, the solutions are of class $\mathcal{C}^{\beta}$.

Step 1, arbitrary dimension. Write $u = (u_1, u')$ with $u' = (u_2, \dots, u_n)$. Recall that $\frac{\partial Q}{\partial\overline{z}_1}(z_1, 0) = 0$. So (E) gives us:

$$\frac{\partial^2\overline{u}_1}{\partial\zeta\partial\overline{\zeta}}(0) =^o \sum_{j\geq 2}\overline{\frac{\partial u_j}{\partial\zeta}}\sum_{k\geq 1}a_{j,k}\frac{\partial u_k}{\partial\zeta} \ , \qquad (E'')$$

and (with a different arrangements of terms)

$$\frac{\partial^2\overline{u}'}{\partial\zeta\partial\overline{\zeta}}(0) =^o \sum_{k\geq 1}\frac{\partial u_k}{\partial\zeta}M_k(u_1)\overline{\frac{\partial u'}{\partial\zeta}} \qquad (E''')$$

where the right hand side is evaluated at $\zeta = 0$, and with

$$a_{j,k}(z_1) = \frac{\partial q_{1,k}}{\partial\overline{z}_j}(z_1, 0, \dots, 0) \ ,$$

and where finally $M_k(z_1)$ is an $(n-1)\times(n-1)$ matrix of class $\mathcal{C}^{\beta-1}$, with entries that are entries of the matrices $\frac{\partial Q}{\partial\overline{z}_j}(z_1, 0, \dots 0)$. The goal is to have the coefficients $a_{j,1} = 0$ and the matrix $M_1 = 0$, and keep $Q = 0$ on the z_1 axis For this, one makes a change of variables, leaving the z_1 axis invariant and preserving J_{st} along the z_1 axis, given by:

$$Z_1 = z_1 + \sum_{j\geq 2} z_j f_j(z_1) \qquad , \qquad Z' = A(z_1)z' \ ,$$

where f_j's are functions and A is an invertible matrix valued function to chose appropriately. To any (germ of) J-holomorphic disc $u=(u_1,u')$ corresponds a J-holomorphic disc $U=(U_1,U')=(U_1,U_2,\dots,U_n)$ with

$$U_1 = u_1 + \sum_{j\geq 2} u_j f_j(u_1)\ ,\ U' = A(u_1)u'\ . \tag{F}$$

We have $\frac{\partial u_j}{\partial \overline{\zeta}} =^o 0$ and $\frac{\partial U_j}{\partial \overline{\zeta}} =^o 0$, and we get

$$\begin{aligned}\frac{\partial U_1}{\partial \zeta} &=^o \frac{\partial u_1}{\partial \zeta} + \sum_{j\geq 2}\frac{\partial u_j}{\partial \zeta} f_j(u_1)\\ \frac{\partial U'}{\partial \zeta} &=^o A(u_1)\frac{\partial u'}{\partial \zeta}\ .\end{aligned}$$

One can solve for $\frac{\partial u}{\partial \zeta}$, in terms of $\frac{\partial U}{\partial \zeta}$. then for approriate functions $b_j(z_1)$, one gets

$$\begin{aligned}\frac{\partial u_1}{\partial \zeta} &=^o \frac{\partial U_1}{\partial \zeta} + \sum_{j\geq 2} b_j(u_1)\frac{\partial U_j'}{\partial \zeta}\\ \frac{\partial u'}{\partial \zeta} &=^o A^{-1}(u_1)\Big(\frac{\partial U'}{\partial \zeta}\Big)\ .\end{aligned}$$

Taking the Laplacian of each side in the (conjugate of the) equations (F) leads to

$$\frac{\partial^2 \overline{U}_1}{\partial\zeta\partial\overline{\zeta}} =^o \frac{\partial^2 \overline{u}_1}{\partial\zeta\partial\overline{\zeta}} + \sum_{j\geq 2}\frac{\partial^2 \overline{u}_j}{\partial\zeta\partial\overline{\zeta}}\overline{f}_j(u_1) + \sum_{j\geq 2}\overline{\frac{\partial u_j}{\partial \zeta}}\frac{\partial \overline{f}}{\partial z_1}(u_1)\frac{\partial u_1}{\partial \zeta}\ .$$

Finally using the expressions given by (E'') and (E''') for the second derivatives, one has, with $c^r_{j,k}(z_1)=\frac{\partial q_{j,k}}{\partial \overline{z}_r}$:

$$\begin{aligned}\frac{\partial^2 \overline{U}_1}{\partial\zeta\partial\overline{\zeta}} =^o &\sum_{j\geq 2}\overline{\frac{\partial u_j}{\partial \zeta}}\sum_{k\geq 1} a_{j,k}(u_1)\frac{\partial u_k}{\partial \zeta}\\ &+\sum_{j\geq 2}\sum_{r\geq 2}\overline{\frac{\partial u_r}{\partial \zeta}}\sum_{k\geq 1} c^r_{j,k}(u_1)\frac{\partial u_k}{\partial \zeta}\overline{f}_j(u_1)\ +\sum_{j\geq 2}\overline{\frac{\partial u_j}{\partial \zeta}}\frac{\partial \overline{f}_j}{\partial z_1}(u_1)\frac{\partial u_1}{\partial \zeta}\ ,\end{aligned}$$

and

$$\frac{\partial^2 \overline{U}'}{\partial\zeta\partial\overline{\zeta}} =^o \overline{A}(u_1)\Big(\sum_{k\geq 1}\frac{\partial u_k}{\partial \zeta}M_k(u_1)\overline{\frac{\partial u'}{\partial \zeta}}\Big)\ +\frac{\partial u_1}{\partial \zeta}\frac{\partial \overline{A}}{\partial z_1}\overline{\frac{\partial u'}{\partial \zeta}}\ .$$

Then, one would have to express the $\frac{\partial u_j}{\partial \zeta}$'s in terms of the $\frac{\partial U_j}{\partial \zeta}$'s using the expressions given above. As previously, having no term with a $\frac{\partial U_1}{\partial \zeta}$ after this substitution is equivalent to having cancellation of the $\frac{\partial u_1}{\partial \zeta}$ factors in the right hand side. In order to have cancellation of the terms with the factor $\frac{\partial u_1}{\partial \zeta}$, we are led to the conditions that the f_j's and A must satisfy, all functions being functions of z_1. (To follow the next lines, it is easier to change notation and permute the indices j and r in the second term on the right hand side of the equation for $\frac{\partial^2 \overline{U}_1}{\partial\zeta\partial\overline{\zeta}}$.)

For the $(n-1)$ functions f_j's (by grouping together the terms with a factor $\overline{\frac{\partial u_j}{\partial \zeta}} \frac{\partial u_1}{\partial \zeta}$), one gets a differential system of $(n-1)$ equations:

$$\frac{\partial \overline{f}_j}{\partial z_1} + \sum_{r\geq 2} c^j_{r,1} \overline{f}_r + a_{j,1} = 0 \qquad (j = 2, \dots, n) \tag{1}$$

and for the matrix A, to be found invertible, one has the equation

$$\frac{\partial \overline{A}}{\partial z_1} + \overline{A} M_1 = 0 \ . \tag{2}$$

These equations can be solved as explained in the appendix, although this is less elementary than in the case of dimension 2, and that ends the proof of Theorem 2.

Remark 2 Said in terms of J, rather than Q, $\partial J(e_1) = 0$ was the goal of Step 1 (see proof of Proposition 1). The goal would be reached if one had $J(e_1) \equiv ie_1$, in which case we would even have $d\ J(e_1) = 0$. That can be done (at least locally) by foliating $\mathbf{C}^n$ by J-holomorphic curves, changing variables to make the leaves parallel to the z_1 axis, and with appropriate parameterizations of the leaves. But such a foliation would not allow to keep the standard structure along the z_1 axis. What is done in Step 1 is far less ambitious. One does not take care of the $\overline{z}$ derivatives, and moreover one gets a result only along the z_1 axis. A more direct geometric approach for Step 1 remains desirable.

Remark 3 In many circumstances one reduces to the case of an almost complex structure J close to the standard struture. For working locally near a point, this is done by simple rescaling ($z \mapsto Kz$, K large positive number). It is worth noticing that such a simple rescaling does not apply when working along a fixed disc, like the unit disc on the z_1 axis. If one tries to use a non isotropic scaling like $z \mapsto (z_1, Kz_2, \dots, Kz_n)$, the matrix Q (assumed to be 0 on the z_1 axis) is replaced by a matrix $\tilde{Q}$ and by simple considerations on the equation for J-holomorphic discs, the coefficients of these matrices are related by:

$$\begin{aligned}
\tilde{q}_{1,1}(z) &= q_{1,1}\Big(z_1, \frac{1}{K}z_2, \dots, \frac{1}{K}z_n\Big) \\
\tilde{q}_{1,r}(z) &= \frac{1}{K} q_{1,r}\Big(z_1, \frac{1}{K}z_2, \dots, \frac{1}{K}z_n\Big), \ \text{for} \ \ r \geq 2 \\
\tilde{q}_{r,1}(z) &= K q_{r,1}\Big(z_1, \frac{1}{K}z_2, \dots, \frac{1}{K}z_n\Big) \ \ \text{for} \ r \ \geq 2 \\
\tilde{q}_{r,s}(z) &= q_{r,s}\Big(z_1, \frac{1}{K}z_2, \dots, \frac{1}{K}z_n\Big) \ \text{for} \ r \ \ \text{and} \ s \geq 2 \ .
\end{aligned}$$

So at any point z on the z_1 axis, for any r and $j \geq 2$ $\frac{\partial \tilde{q}_{r,1}}{\partial \overline{z}_j} = \frac{\partial q_{r,1}}{\partial \overline{z}_j}$ ($= c^j_{r,1}$ in the proof). Consequently the coefficients $c^j_{r,1}$ in the proof, are not made to be small, as it would be desirable.

4. Appendix

(Changing notations). Solvability of the equations (1) and (2) at the end of the proof of Theorem 2 follows from:

($) *Let $k \in \mathbf{N}$, and $0 < \alpha < 1$. If $\zeta \mapsto M(\zeta)$ is a continuous function defined on a neighborhood of the unit disc in $\mathbf{C}$, with values in the space $M_n(\mathbf{C})$ (of $(n \times n)$ matrices with complex coefficients), then there exists a continuous matrix valued map $\zeta \mapsto A(\zeta)$ defined on a neighborhood of the unit disc, with values in $GL(n, \mathbf{C})$ (the set of invertible matrices), such that*

$$\frac{\partial A}{\partial \bar{\zeta}} + A(\zeta)M(\zeta) = 0 \ .$$

If M is of class $\mathcal{C}^{k,\alpha}$, for some $k \in \mathbf{N}$ and $0 < \alpha < 1$, A is of class $\mathcal{C}^{k+1,\alpha}$. This applies to the equation $\frac{\partial A}{\partial \bar{\zeta}} + MA = 0$, by transposition.

This result gives the solution of (2). For (1), set $f = (f_2, \ldots, f_n)$ and (with z_1 being now ζ), rewrite (1) in the form

$$\frac{\partial f}{\partial \bar{\zeta}} + C(\zeta)f + g(\zeta) = 0 \ .$$

Let A be an invertible matrix valued function satisfying $\frac{\partial A}{\partial \bar{\zeta}} + CA = 0$. The variation of parameters method consisiting in setting $f = Ah$, leads to the simple equation $\frac{\partial h}{\partial \bar{\zeta}} + A^{-1}g = 0$.

Note that one does not need to use results on regularity up to the boundary since one can as well start by extending the data. Regularity is then immediate by induction, knowing that if a function ψ in one complex variable is such that $\frac{\partial \psi}{\partial \bar{\zeta}}$ is of class $\mathcal{C}^\gamma$ (for $\gamma = 0$, bounded), then ψ is of class $\mathcal{C}^{\gamma+1}$ unless γ is an integer, in which case one gets only $\mathcal{C}^{\gamma'}$ regularity for $\gamma' < \gamma + 1$.

The result ($) is due to Malgrange (who gives a version with parameters) [6, Theorem 1 in Chapter IX]. Here we sketch a proof. First we look at the case when M has small sup-norm, and sup-norm will be used throughout in the argument. Let T_{CG} denote the Cauchy-Green operator acting on functions (possibly vector or matrix valued) φ defined on the closed unit disc defined by

$$T_{CG}\varphi(\zeta) = \frac{1}{\pi} \int_{\mathbf{D}} \frac{\varphi(\eta)}{\zeta - \eta} \, dxdy(\eta) \, .$$

Its basic property is that $\frac{\partial}{\partial \bar{\zeta}}(T_{CG}\varphi) = \varphi$. Set $A = \mathbf{1} + B$. The equation to solve becomes $\frac{\partial B}{\partial \bar{\zeta}} + BM + M = 0$. This can be written as

$$\frac{\partial}{\partial \bar{\zeta}}(B + T_{CG}BM + T_{CG}M) \ ,$$

which can be solved by asking $B + T_{CG}BM + T_{CG}M = 0$. If M has small sup-norm the operator $B \mapsto B + T_{CG}BM$ is a small perturbation of the identity, and the equation

$B + T_{CG}BM + T_{CG}M = 0$ has a small solution, giving us A invertible, defined on the unit disc. The same applies to any disc, with the smallness of M depending on the radius, by simple rescaling. The smaller the disc, the larger M is allowed to be. So, in case of large M the above argment provides only local solutions, say A_j. Note that if A is a solution, RA is a solution if and only if R is holomorphic. So the local solutions differ by holomorphic invertible factors, i.e. $A_j A_k^{-1} = C_{j,k}$ (defined on the intersection of their domains) where $C_{j,k}$ is holomorphic. The Cartan Lemma on holomorphic matrices allows one to write $C_{j,k} = H_j H_k^{-1}$ with holomorphic matrices, leading to the global solution given by the $H_j^{-1}A_j$'s.

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