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A Geometrical approach to Gordan–Noether's and Franchetta's contributions to a question posed by Hesse

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Abstract

Hesse claimed in [7] (and later also in [8]) that an irreducible projective hypersurface in $\mathbb{P}^n$ defined by an equation with vanishing hessian determinant is necessarily a cone. Gordan and Noether proved in [6] that this is true for $n \leq 3$ and constructed counterexamples for every $n \geq 4$. Gordan and Noether and Franchetta gave classification of hypersurfaces in $\mathbb{P}^4$ with vanishing hessian and which are not cones, see [6, 5]. Here we translate in geometric terms Gordan and Noether approach, providing direct geometrical proofs of these results.

Introduction

Let $f = f(x_0, \dots, x_n) \in k[x_0, \dots, x_n]$ be a non-zero irreducible homogeneous polynomial over an algebraically closed field k of characteristic zero. Then the hessian polynomial of f is the determinant of the matrix of the second partial derivatives:

$$h_f := \det([\partial^2 f / \partial x_i \partial x_j]_{i,j=0,\dots,n}).$$

Obviously if the hypersurface $X = V(f) \subset \mathbb{P}^n$ is a cone (i.e. up to a linear change of coordinates f does not depend on all the variables), then the hessian polynomial

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of f is identically zero. The converse is clearly true when $\deg(f) \leq 2$. Hesse claimed twice that the converse is true also for $\deg(f) \geq 3$, i.e. he claimed that if the hessian polynomial of a polynomial f of degree at least three is identically zero then the hypersurface $X = V(f) \subset \mathbb{P}^n$ is a cone (see [7, 8]).

The problem was reconsidered by Gordan and Noether [6] who proved that Hesse's claim is true when $n \leq 3$ but false in general when $n \geq 4$. They constructed families of counterexamples for every $n \geq 4$, which have been revisited later by Permutti in [11, 12] and more recently by Lossen in [9]. Moreover, Gordan and Noether seem to have proved that their families of examples are the only possible counterexamples if $n = 4$ but it is rather difficult to indicate a precise reference for this result in their monumental paper. Franchetta [5] gave an independent classification of hypersurfaces in $\mathbb{P}^4$ with vanishing hessian which are not cones using more geometrical techniques. Other examples were given by Perazzo [10], who considered the case of cubic hypersurfaces with vanishing hessian and obtained the classification of these cubics in $\mathbb{P}^4$, $\mathbb{P}^5$ and $\mathbb{P}^6$.

Since the problem posed by Hesse has a geometrical flavour, the aim of this note is to translate Gordan and Noether's approach in more geometric terms, using some ideas and results contained in [6, 9] and in the recent [1]. We also briefly describe the counterexamples in projective spaces of dimension at least four produced by Gordan and Noether, relating them to work of Franchetta and Permutti and we provide a short geometrical proof of the characterization of hypersurfaces in $\mathbb{P}^4$ with vanishing hessian which are not cones.

In the first Section we describe some background material and we consider a geometrical construction involving the dual variety of a hypersurface. This construction allows us to reconsider Gordan and Noether's results and to describe them in a geometrical context. In the second Section Hesse's claim is proved in the case of hypersurfaces of dimension at most 2. This proof is very easy and it is based on the geometrical construction given in the first Section. In the third Section the counterexamples by Gordan and Noether and Franchetta are described, using also results from [11, 12, 1]. The last Section is dedicated to hypersurfaces in $\mathbb{P}^4$. We describe the properties of hypersurfaces in $\mathbb{P}^4$ with vanishing hessian and then we give a new proof of Franchetta's classification of these hypersurfaces.

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1. Background material

1.1 The Polar map and the Hessian of a projective hypersurface

Consider a non-constant homogeneous polynomial of degree $d \geq 1$, $f = f(x_0, \ldots, x_n) \in k[x_0, \ldots, x_n]$, in the $n+1$ variables $x_0, \ldots, x_n$ over an algebraically closed field k of characteristic zero. Denote by f_i the partial derivatives $\partial f / \partial x_i$, $i = 0, \ldots, n$.

Definition 1.1 Let $X = V(f) \subset \mathbb{P}^n$ be the associated hypersurface. We say that X is a *cone* if, modulo projective transormations of $\mathbb{P}^n$, the equation defining X does not depend on all the variables.

Equivalently X is a cone if and only if $\mathrm{Vert}(X) \neq \varnothing$, where $\mathrm{Vert}(X)$, the *vertex of* X, is the set:

$$\mathrm{Vert}(X) := \{x \in X : \quad J(x, X) = X\},$$

and

$$J(x, X) = \overline{\bigcup_{y \neq x, y \in X} \langle x, y\rangle} \subset \mathbb{P}^n$$

is *the join of* x *and* X.

We recall that, if $X \subset \mathbb{P}^n$ is an (irreducible) subvariety of $\dim(X) = d$, then

$$\mathrm{Vert}(X) = \bigcap_{x \in X} T_x X = \mathbb{P}^l \subset X,$$

with $l \geq -1$ (see e.g. [13, Proposition 1.2.6]).

Definition 1.2 The (*first*) *polar map* associated to the hypersurface $X = V(f) \subset \mathbb{P}^n$ is the rational map $\phi_f : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$, defined by the partial derivatives of f:

$$\phi_f(p) = (f_0(p) : \ldots : f_n(p)).$$

If $p \in X = V(f)$ is not singular, the polar map ϕ_f can be interpreted as mapping the point $p \in X$ to its tangent hyperplane T_pX (and, as such, the target of the map ϕ_f is $\mathbb{P}^{n*}$). Note that the base locus of ϕ_f is the scheme $\mathrm{Sing}(X) = V(f_0, \ldots, f_n) \subset \mathbb{P}^n$. Denote by $Z(f) \subset \mathbb{P}^{n*}$ the closure of the image of $\mathbb{P}^n$ under the polar map ϕ_f. The variety $Z(f) \subset \mathbb{P}^{n*}$ is called the *polar image* of f.

Definition 1.3 We define the *Hessian matrix* of the polynomial f to be the $(n+1) \times (n+1)$ matrix:

$$\mathcal{H}_f := \left(\frac{\partial^2 f}{\partial x_i \partial x_j}\right)_{i,j=0,\ldots,n}.$$

Its determinant $h_f := \det(\mathcal{H}_f) \in k[x_0, \ldots, x_n]$ is the *Hessian polynomial* of f.

Note that the Jacobian matrix J_{ϕ_f} of the (affine) polar map ϕ_f is exactly the Hessian matrix of f, $\mathcal{H}_f$.

We recall now the construction of the dual variety X^* of an algebraic reduced variety $X \subset \mathbb{P}^n$.

Let $\mathrm{Sm}(X)$ denote the open non-empty subset of non singular points of a reduced variety $X \subset \mathbb{P}^n$. Let

$$\mathcal{P}_X := \overline{\{(x, H) : x \in \mathrm{Sm}(X), T_xX \subset H\}} \subset X \times \mathbb{P}^{n*}$$

be the *conormal variety* of X, and consider the projections of $\mathcal{P}_X$ onto the factors:

$$\begin{tikzcd} & \mathcal{P}_X \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ X & & X^* = \overline{p_2(\mathcal{P}_X)} \subset \mathbb{P}^{n*}. \end{tikzcd}$$

The dual variety of X, X^*, is the scheme-theoretic image of $\mathcal{P}_X$ in $\mathbb{P}^{n^*}$. In particular if $X \subset \mathbb{P}^n$ is a hypersurface, then $X^* \subset \mathbb{P}^{n*}$ is the closure of the set of hyperplanes tangent to X at non–singular points. Observe that since the Gauss map of X associates to a non singular point $p \in X$ the point in $\mathbb{P}^{n*}$ corresponding to the hyperplane tangent to X in p, we infer that (when X is a hypersurface) the closure of the image of the Gauss map of X is exactly the dual variety X^*.

Note also that the restriction of the polar map ϕ_f to $V(f) \setminus \mathrm{Sing}(V(f))$ is the Gauss map of $X = V(f)$, hence the closure of the image of X via ϕ_f is the dual variety X^* of X.

1.2 Hypersurfaces with vanishing Hessian

We recall that $f_0, \dots, f_r \in k[x_0, \dots, x_r]$ are *algebraically dependent* if there exists a non zero polynomial

$$\pi = \pi(y_0, \dots, y_r) \in k[y_0, \dots, y_r]$$

such that $\pi(f_0, \dots, f_r) = 0$. In particular they are linearly dependent if and only if there exists such a π of degree one.

Note first that the following easy fact holds, recalling that the Jacobian matrix of the affine polar map $\widehat{\phi_f} : k^{n+1} \dashrightarrow k^{n+1}$ is the hessian matrix $\mathcal{H}_f$.

Proposition 1.4

Let $f \in k[x_0, \dots, x_n]$ be a homogeneous polynomial. Then the following conditions are equivalent:

- $h_f \equiv 0$;
- ϕ_f *is not a dominant map;*
- $Z(f) \subsetneq \mathbb{P}^{n*}$;
- $f_0, \dots, f_n$ *are algebraically dependent.*

We recall the following result from [2], which proves a conjecture stated in [3].

Proposition 1.5 [2, Corollary 2]

The degree of the polar map ϕ_f depends only on $\mathrm{Supp}(V(f))$*, where, by definition, the degree of ϕ_f is zero if and only if ϕ_f is not a dominant map.*

Note that by Proposition 1.4 the property of having vanishing Hessian is equivalent to the fact that $\dim(Z(f)) < n$, whence by Theorem 1.5 this property depends only on the support of the hypersurface $X = V(f)$.

Since we are interested in hypersurfaces with vanishing Hessian, from now on we shall assume that $X = V(f)$ is a reduced (and irreducible) hypersurface. The following result is due to Zak (see [14, Proposition 4.9]).

Proposition 1.6

Let $X = V(f) \subset \mathbb{P}^n$ be a reduced hypersurface with vanishing Hessian and let $Z(f) \subsetneq \mathbb{P}^{n*}$ denote the polar image of f. Suppose $d \geq 2$, i.e. ϕ_f not constant. Then:

$$Z(f)^* \subset \mathrm{Sing}(X).$$

In particular, Sing(X) $\neq \emptyset$, $\dim(Z(f)^*) \leq n - 2$ *and* $X^* \subsetneq Z(f)$.

In the sequel we shall need the following well-known result, see for example [4, Proposition 1.1].

Proposition 1.7

The hypersurface $V(f) = X$ is a cone if and only if X^* is a degenerate variety. In particular the hypersurface $X = V(f)$ is a cone if and only if the partial derivative of f are linearly dependent.

Now we recall a problem considered twice by Hesse in [7, 8], giving an equivalent geometric formulation of it. Note that obviously, if $X = V(f) \subset \mathbb{P}^n$ is a cone, i.e. up to a linear change of variables f does not depend on all the variables, then $h_f \equiv 0$. One can ask if the converse holds, i.e. if

$$\text{Hesse's claim: } \textit{if } h_f \equiv 0\textit{, then } V(f) \subset \mathbb{P}^n \textit{ is a cone}$$

is always true.

Note that by Proposition 1.7 Hesse's claim is equivalent to the following: $h_f \equiv 0$ if and only if the derivatives of f are linearly dependent.

The question was reconsidered by Gordan and Noether in [6]. They showed that Hesse's claim is true when $n \leq 3$ but it is false in general when $n \geq 4$. They constructed families of counterexamples which have been revisited by Permutti in [11, 12] and by Lossen in [9]. An easy example for $n = 4$ is the following cubic polynomial $f(x_0, x_1, x_2, x_3, x_4) = x_0x_3^2 + 2x_1x_3x_4 + x_2x_4^2$.

Remark 1.8 Note that if $d = \deg(f) \leq 2$ then Hesse's claim is true for every $n \geq 1$. Indeed if $d = 1$ then $V(f)$ is a hyperplane, and so it is a cone and $V(f)^*$ is a point. If $d = 2$ then $V(f)$ is a hyperquadric and $\mathcal{H}_f$ is the matrix associated to the quadratic form of $V(f)$. Since its determinant is zero, the associated hyperquadric is singular, and so the hyperquadric is a cone.

From now on $f \in k[x_0, \ldots, x_n]$ will be a homogeneous reduced polynomial of degree $d \geq 3$ such that $h_f \equiv 0$, unless otherwise stated.

Since $h_f \equiv 0$, there exist non zero homogeneous polynomials $\pi \in k[y_0, \ldots, y_n]$ such that $\pi(f_0, \ldots, f_n) \in k[x_0, \ldots, x_n]$ is identically equal to zero. Let $g \in k[y_0, \ldots, y_n]$ be such a polynomial and such that

$$g_i := \frac{\partial g}{\partial y_i}\Big(\frac{\partial f}{\partial x_0}, \ldots, \frac{\partial f}{\partial x_n}\Big) \in k[x_0, \ldots, x_n],\ i = 0, \ldots, n\,,$$

are not all identically equal to zero.

Definition 1.9 Let $S = V(g) \subset \mathbb{P}^{n*}$ be an irreducible and reduced hypersurface containing the polar image $Z(f)$, where $g(y_0, \ldots, y_n)$ is as above. By definition of g the variety $Z(f)$ is not completely contained in the singular locus of S. Let

$$\psi_g \colon \mathbb{P}^n \dashrightarrow \mathbb{P}^n$$

be the composition of ϕ_f with ϕ_g (or equivalently ψ_g is the composition of ϕ_f with the Gauss map of S). If the polynomials

$$g_i := \frac{\partial g}{\partial y_i}\left(\frac{\partial f}{\partial x_0}, \ldots, \frac{\partial f}{\partial x_n}\right) \in k[x_0, \ldots, x_n]$$

have a common divisor $\rho := \text{g.c.d.}(g_0, \ldots, g_n) \in k[x_0, \ldots, x_n]$, set

$$h_i := \frac{g_i}{\rho} \in k[x_0, \ldots, x_n], \text{ for } i = 0, \ldots, n\,.$$

It follows that the map ψ_g is given by:

$$\psi_g(p) = \Big(g_0\big(f_0(p), \ldots, f_n(p)\big) : \ \ldots : g_n\big(f_0(p), \ldots, f_n(p)\big)\Big) = \big(h_0(p) : \ \ldots : h_n(p)\big),$$

with $\text{g.c.d.}(h_0, \ldots, h_n) = 1$.

So we have:

$$\begin{array}{ccccc}
X & \overset{\mathcal{G}_X}{\dashrightarrow} & X^* & & \\
\cap & & \cap & & \\
\mathbb{P}^n & \overset{\phi_f}{\dashrightarrow} & Z(f) \subset S = V(g) \subset \mathbb{P}^{n*} & \overset{\phi_g}{\dashrightarrow} & \mathbb{P}^{n**} \cong \mathbb{P}^n \\
& & & & \\
\mathbb{P}^n & & \overset{\psi_g = \phi_g \circ \phi_f}{\dashrightarrow} & & \mathbb{P}^{n**} \cong \mathbb{P}^n.
\end{array}$$

Set $S^*_Z := \overline{\psi_g(\mathbb{P}^n)}$ and note that by definition of ψ_g,

$$S^*_Z \subset Z(f)^*.$$

Indeed $Z(f)^*$ consists of the hyperplanes containing the tangent spaces to $Z(f)$, while S^*_Z consists of the hyperplanes which are tangent to S at points of $Z(f)$. Since $Z(f) \subset S$ the hyperplanes which are tangent to S in points of $Z(f)$ are clearly hyperplanes containing the tangent spaces to $Z(f)$.

Recalling Proposition 1.6, we get:

$$S^*_Z \subset Z(f)^* \subset \text{Sing}(X). \tag{1}$$

Remark 1.10 Note that the base locus of ψ_g is the scheme $\text{Bs}(\psi_g) = V(h_0, \ldots, h_n) \subset \mathbb{P}^n$. Let $e = \deg(h_i) \geq 0$. By definition of $S = V(g)$ there exists at least one i such that $g_i \neq 0$ so that there exists at least one i such that $h_i \neq 0$. In particular $\dim(S^*_Z) \geq 0$ with equality holding if and only if $e = 0$ or all the h_i's are zero except one.

Clearly $\dim(S^*_Z) = 0$ if and only if $Z(f)$ is contained in the hyperplane $(S^*_Z)^*$ if and only if the partial derivatives of f are linearly dependent if and only if $X = V(f)$ is a cone.

If $\dim(S^*_Z) \geq 1$, then at least two h_i's are non-zero and $e \geq 1$, so that $\dim(V(h_0, \ldots, h_n)) \leq n - 2$.

Let us recall a fundamental result proved by Gordan and Noether (see [6] and [9, (2.7)]).

Theorem 1.11

Let notation be as above and let $F \in k[x_0, \dots, x_n]$. Then:

$$\sum_{i=0}^{n} \frac{\partial F}{\partial x_i} h_i = 0 \Leftrightarrow F(\underline{x}) = F(\underline{x} + \lambda \psi_g(\underline{x})) \ \forall \lambda \in k. \tag{2}$$

Remark 1.12 Note that $\sum_{i=0}^{n} \frac{\partial^2 f}{\partial x_i \partial x_j} h_i = 0$. This relation is obtained differentiating the equation $g(f_0, \dots, f_n) = 0$ with respect to x_j and applying the chain rule. As a consequence, we get the following relation by Theorem 1.11:

$$f_i(\underline{x}) = f_i(\underline{x} + \lambda \psi_g(\underline{x})). \tag{3}$$

Remark 1.13 Using the above result one can prove $\sum_{i=0}^{n} \frac{\partial g_k}{\partial x_i} h_i = 0$.

Indeed since g_k is a polynomial in $\frac{\partial f}{\partial x_j}$, $j = 0, \dots, n$,

$$\sum_{i=0}^{n} \frac{\partial g_k}{\partial x_i} h_i = \sum_{i=0}^{n} \left(\sum_{j=0}^{n} \frac{\partial g_k}{\partial (\frac{\partial f}{\partial x_j})} \cdot \left(\frac{\partial^2 f}{\partial x_j \partial x_i} \right) \right) h_i = 0,$$

where the last equality follows from Remark 1.12.

Remark 1.14 If either one of the two equivalent conditions in (2) holds for a polynomial $F \in k[x_0, \dots, x_n]$, then:

$$S_Z^* \subset V(F).$$

Indeed, using equation (2) and applying Taylor's formula we have:

$$0 = F(\underline{x}) - F\big(\underline{x} + \lambda \psi_g(\underline{x})\big) = \sum_{k=1}^{e} \lambda^k \Phi_k,$$

with $e = \deg(F)$, $\Phi_k := \sum_{i_1 \dots i_k} \frac{\partial^k F}{\partial x_{i_1} \dots \partial x_{i_k}} \frac{h_{i_1} \dots h_{i_k}}{k!}$. In particular, if we assume $F \neq 0$ and homogeneous of degree $e \geq 1$, comparing the coefficient for λ^e we get: $F(\psi_g(\underline{x})) = F(h_0, \dots, h_n) = 0$.

Remark 1.15 Let $F = A \cdot B$ where $A, B, F \in k[x_0, \dots, x_n]$. If either one of the two equivalent conditions in (2) holds for F then the same condition holds for A and B ([9, (2.7)]).

Since the property holds for the polynomials $g_k = \rho h_k$, it holds also for the polynomials h_k and hence

$$\sum_{i=0}^{n} \frac{\partial h_k}{\partial x_i} h_i = 0. \tag{4}$$

Since $\psi_g = (h_0 : \dots : h_n)$, by Theorem 1.11 we have (cf. [9, (2.8)])

$$\forall p \in \mathbb{P}^n, \forall \lambda \in k, \ \psi_g(p) = \psi_g(p + \lambda \psi_g(p)). \tag{5}$$

Remark 1.16 Geometrically, equation (5) means that the fiber of ψ_g over a point $q \in S_Z^*$, $\psi_g^{-1}(q)$, is a cone whose vertex contains the point q. Indeed $\forall p \in \mathbb{P}^n$ such that $\psi_g(p) = q$, $q = \psi_g(p) = \psi_g(p + \lambda\psi_g(p)) = \psi_g(p + \lambda q)$, i.e. $p + \lambda q \in \psi_g^{-1}(q)$ for all λ. Hence $\forall p \in \mathbb{P}^n$ such that $\psi_g(p) = q$, the line $\langle p, q\rangle$ is contained in $\psi_g^{-1}(q)$.

Collecting the above remarks we get the following result.

Proposition 1.17

Let notation and hypothesis be as above. Then:

$$S_Z^* \subset V(h_0, \dots, h_n) = \mathrm{Bs}(\psi_g).$$

Proof. By equation (4), $\sum_{i=0}^n \frac{\partial h_k}{\partial x_i} h_i = 0$, hence one of the two equivalent conditions in (2) holds for h_i, $i = 0, \dots n$. By Remark 1.14 this implies that $S_Z^* \subset V(h_0, \dots, h_n) = Bs(\psi_g)$. □

We have the following useful proposition will be used later.

Proposition 1.18

Let notation be as above, let $q \in S_Z^*$ be a general point and let $w \in \mathrm{Bs}(\psi_g)$ (respectively $t \in \mathrm{Sing}(X)$). If $w \in \overline{\psi_g^{-1}(q)} \setminus \{q\}$, (respectively $t \in \overline{\psi_g^{-1}(q)} \setminus \{q\}$), then the line $\langle w, q\rangle$ is contained in $\mathrm{Bs}(\psi_g)$, (respectively the line $\langle t, q\rangle$ is contained in $\mathrm{Sing}(X)$).

Proof. Since $\psi_g^{-1}(q)$ is a cone whose vertex contains the point q by (5), then $\overline{\psi_g^{-1}(q)}$ is a cone whose vertex contains the point q. The line $\langle w, q\rangle$ (respectively the line $\langle t, q\rangle$) is contained in $\overline{\psi_g^{-1}(q)}$, whence the conclusion follows from relation (5) (respectively (3)). □

Remark 1.19 For each $q \in S_Z^*$ and for each $p \in \mathbb{P}^n$ such that $\psi_g(p) = q$, we have $\langle p, q\rangle \cap \mathrm{Bs}(\psi_g) = \{q\}$ as sets. Indeed let us suppose that there exists $r \neq q$ such that $r \in \langle p, q\rangle \cap \mathrm{Bs}(\psi_g)$. Then $r = p + \lambda q$ and $\psi_g(r) = \psi_g(p)$ by (5). Since $r \in \mathrm{Bs}(\psi_g)$, we would deduce $p \in \mathrm{Bs}(\psi_g)$, contrary to our assumption.

Another general and useful remark is the following lemma which gives a connection between the polar map of the restriction to a hyperplane with the geometry of $Z(f)$ (see [1, Lemma 3.10]).

Lemma 1.20

Let $X = V(f) \subset \mathbb{P}^n$ be a hypersurface. Let $H = \mathbb{P}^{n-1}$ be a hyperplane not contained in X, let $h = H^*$ be the corresponding point in $\mathbb{P}^{n*}$ and let π_h denote the projection from the point h. Then:

$$\phi_{V(f)\cap H} = \pi_h \circ (\phi_{f|H}).$$

In particular, $Z(V(f) \cap H) \subset \pi_h(Z(f))$, where $Z(V(f) \cap H)$ denotes the closure of the image of the polar map $\phi_{V(f)\cap H}$ of the hypersurface $V(f) \cap H$ of H.

2. Cases in which Hesse's claim is true

In this section we shall consider some hypotheses under which the conclusion in Hesse's claim holds.

Remark 2.1

i) Let $S = V(g) \supseteq Z(f)$. If S^* is a cone, then $X = V(f)$ is a cone. If $Z(f)^*$ is a cone then X is a cone.

   Indeed if S^* (resp. $Z(f)^*$) is a cone, then S (resp. $Z(f)$) is a degenerate variety of $(\mathbb{P}^n)^*$. Since $X^* \subset Z(f) \subset S$, X^* is a degenerate variety, whence X is a cone.

ii) In particular if $\dim(S^*) = 0$ (resp. $\dim(Z(f)^*) = 0$) then X is a cone.

iii) If $\dim(S^*_Z) = 0$, then X is a cone (cf. Remark 1.10).

We also recall some properties of the cone X which are described dually by other geometric properties of its dual variety X^*.

Remark 2.2

i) If X^* is a non-degenerate subvariety of a linear subspace $L = \mathbb{P}^{n-m}$ $(m = 1, \ldots, n-1)$ in $\mathbb{P}^{n*}(\cong \mathbb{P}^n)$, then X is a cone with vertex a linear subspace $\mathbb{P}^{m-1} = L^*$.

ii) If X^* is union of $d \geq 1$ points which span a linear subspace $\mathbb{P}^{n-m}$ of $(\mathbb{P}^n)^*$, then X is made up of d hyperplanes whose intersection is a $(m-1)$-linear subspace of $\mathbb{P}^n$.

Now we can easily prove Hesse's claim when $n \leq 3$.

Proposition 2.3

Let $X = V(f) \subset \mathbb{P}^1$ be a reduced hypersurface of degree d. Then $X = V(f)$ has vanishing Hessian if and only if X is a cone. In this case $d = 1$ and X is a point.

Proof. In this case $Z(f) \subsetneq \mathbb{P}^1$ must be a point because ϕ_f is not dominant, so the partial derivatives of f are constant and $d = 1$ since X is reduced, i.e. X is a point. $\square$

Proposition 2.4

Let $X = V(f) \subset \mathbb{P}^2$ be a reduced hypersurface of degree $d \geq 2$. Then $X = V(f)$ has vanishing Hessian if and only if X is a cone, i.e. if and only if X consists of d distinct lines through a point.

Proof. Note that $\dim(Z(f)) \leq 1$. As in Proposition 2.3, $Z(f)$ is a point if and only if $d = 1$. Assume $\dim(Z(f)) = 1$. By Proposition 1.6, $Z(f)^* \subset \mathrm{Sing}(X)$. Since we are assuming X to be reduced, we infer that $Z(f)^*$ is a point, so $Z(f)$ is a line and the hypersurface X is a cone, consisting of d lines meeting in the point $Z(f)^*$ (where d is the degree of f). $\square$

The following result was proved by Gordan and Noether in [6]. Here we give an easier and more geometrical proof of it.

Proposition 2.5

Let $X = V(f) \subset \mathbb{P}^3$ be a reduced hypersurface of degree $d \geq 3$. Then $X = V(f)$ has vanishing Hessian if and only if X is a cone. More precisely, $X = V(f)$ has vanishing Hessian if and only if either X is a cone over a curve with vertex a point or X consists of d distinct planes through a line. In the first case $Z(f)$ is a plane in $\mathbb{P}^{3*}$ while in the second case it is a line in $\mathbb{P}^{3*}$.

Proof. In this case $\dim(S^*_Z) \leq \dim(Z(f)^*) \leq 1$ by (1) and Proposition 1.6.

If $\dim(S^*_Z) = 0$, then X is a cone by Remark 1.10. In particular if also $\dim(Z(f)^*) = 0$, then $Z(f)$ is a plane and hence X is a cone over a curve (which is the dual of the curve X^* with respect to the plane $Z(f)$) with vertex a point (the dual of the plane $Z(f)$). The case $\dim(S^*_Z) = 0$ and $\dim(Z(f)^*) = 1$ is very similar to the case $\dim(S^*_Z) = \dim(Z(f)^*) = 1$ described in the following and hence in this case X consists of d distinct planes through a line.

Assume now that $\dim(S^*_Z) = 1$, yielding $\dim(Z(f)^*) = 1$. Since $Z(f)^*$ and S^*_Z are irreducible, $Z(f)^* = S^*_Z$. Let s_1, s_2 be two distinct general points of S^*_Z. Then $\overline{\psi_g^{-1}(s_i)}$ is a surface which is a cone whose vertex contains the point s_i. Let $t \in \overline{\psi_g^{-1}(s_1)} \cap \overline{\psi_g^{-1}(s_2)} \subset \mathrm{Bs}(\psi_g)$. By Proposition 1.18, the lines $\langle s_i, t\rangle$, $i = 1, 2$, are contained in the base locus of ψ_g. Since $\dim S^*_Z = 1$, by Remark 1.10, $\dim \mathrm{Bs}(\psi_g) \leq 1$. Hence the irreducible component of $\mathrm{Bs}(\psi_g)$ passing through s_1 is exactly the line $\langle s_1, t\rangle$. But also S^*_Z is an irreducible component of $Bs(\psi_g)$ of dimension one passing through s_1, so it has to coincide with the line $\langle s_1, t\rangle$. We conclude that $S^*_Z = Z(f)^* = \langle s_i, t\rangle = \langle s_1, s_2\rangle$. Since $Z(f)^*$ is a line, then $Z(f)$ is a line and $X^* \subsetneq Z(f) = \mathbb{P}^1$, whence X is the union of d planes through $Z(f)^* = \mathbb{P}^1$ by Remark 2.2. □

Corollary 2.6

Let $X = V(f) \subset \mathbb{P}^n$, $n \geq 4$ be a reduced hypersurface of degree d. If $X = V(f)$ has vanishing Hessian and if $\dim(Z(f)) \leq 2$, then $X = V(f)$ is a cone.

Proof. Let $H \subset \mathbb{P}^n$ be a general $\mathbb{P}^3$ and let $h = H^* = \mathbb{P}^{n-4}$. By iterating Lemma 1.20 we deduce that the variety $Z(V(f) \cap H)$ is contained in the variety $\pi_h(Z(f))$, whose dimension equals $\dim(Z(f))$. Thus $V(f) \cap H$ has vanishing Hessian because the polar map $\phi_{V(f)\cap H} \colon \mathbb{P}^3 \dashrightarrow \mathbb{P}^{3*}$ is not dominant. By Proposition 2.5 we infer that $V(f) \cap H$ is a cone. By the generality of H we get that $X = V(f) \subset \mathbb{P}^n$ is a cone. □

3. Gordan–Noether's and Franchetta's counterexamples to Hesse's conjecture

In this section we will describe some examples of hypersurfaces in $\mathbb{P}^n$, $n \geq 4$, with vanishing Hessian and which are not cones, following [6] and [1, §2.3]. Moreover we introduce the hypersurfaces in $\mathbb{P}^4$ which are counterexamples to Hesse's claim described by Franchetta (cf. [5]). We observe that these hypersurfaces are particular cases of the ones described by Gordan–Noether. We also briefly recall the results of Gordan–Noether and Permutti in connection with Hesse's claim, following [1].

Assume $n \geq 4$ and fix integers $t \geq m + 1$ such that $2 \leq t \leq n - 2$ and $1 \leq m \leq n - t - 1$. Consider forms $h_i(y_0, \dots, y_m) \in k[y_0, \dots, y_m]$, $i = 0, \dots, t$, of the same degree, and also forms $\psi_j(x_{t+1},\ldots,x_n) \in k[x_{t+1},\ldots,x_n]$, $j = 0,\ldots,m$, of the same degree. Introduce the following homogeneous polynomials all of the same degree:

$$Q_\ell(x_0,\ldots,x_n) := \det \begin{pmatrix} x_0 & \ldots & x_t \\ \frac{\partial h_0}{\partial \psi_0} & \ldots & \frac{\partial h_t}{\partial \psi_0} \\ \ldots & \ldots & \ldots \\ \frac{\partial h_0}{\partial \psi_m} & \ldots & \frac{\partial h_t}{\partial \psi_m} \\ a^{(\ell)}_{1,0} & \ldots & a^{(\ell)}_{1,t} \\ \ldots & \ldots & \ldots \\ a^{(\ell)}_{t-m-1,0} & \ldots & a^{(\ell)}_{t-m-1,t} \end{pmatrix}$$

where $\ell = 1,\ldots,t-m$. Note that $a^{(\ell)}_{u,v} \in k$ for $u = 1,\ldots,t-m-1$, $v = 0,\ldots,t$ and $\frac{\partial h_i}{\partial \psi_j}$ stands for the derivative $\frac{\partial h_i}{\partial y_j}$ computed at $y_j = \psi_j(x_{t+1},\ldots,x_n)$ for $i = 0,\ldots,t$ and $j = 0,\ldots,m$. Let s denote the common degree of the polynomials Q_ℓ. Taking Laplace expansion along the first row, one has an expression of the form:

$$Q_\ell = M_{\ell,0}x_0 + \ldots + M_{\ell,t}x_t,$$

where $M_{\ell,i}$, $\ell = 1,\ldots,t-m$, $i = 0,\ldots,t$ are homogeneous polynomials of degree $s-1$ in $x_{t+1},\ldots,x_n$.

Fix an integer $d > s$ and set $\mu = [\frac{d}{s}]$. Fix biforms $P_k(z_1,\ldots,z_{t-m};x_{t+1},\ldots,x_n)$ of bidegree $k, d-ks$, $k = 0,\ldots,\mu$. Finally set

$$f(x_0,\ldots,x_n) := \sum_{k=0}^{\mu} P_k(Q_1,\ldots,Q_{t-m},x_{t+1},\ldots,x_n), \tag{6}$$

a form of degree d in $x_0,\ldots,x_n$. The polynomial f is called a *Gordan–Noether polynomial* (or a *GN–polynomial*) of type (n,t,m,s), and so will also any polynomial which can be obtained from it by a projective change of coordinates. Accordingly, a *Gordan–Noether hypersurface* (or a *GN–hypersurface*) of type (n,t,m,s) is the hypersurface $V(f)$, where f is a GN–polynomial of type (n,t,m,s).

The main point of the Gordan–Noether construction is that a GN–polynomial has vanishing Hessian. For a proof see [1, Proposition 2.9]. Another proof closer to Gordan–Noether's original approach is contained in [9].

Proposition 3.1

Every GN–polynomial has vanishing Hessian.

Following [12, 1] we give a geometric description of a GN–hypersurface of type (n,t,m,s) as follows. The main result is that the GN–hypersurfaces have vanishing Hessian (cf. Proposition 3.1) but in general they are not cones, so they are counterexample to Hesse's conjecture.

DEFINITION 3.2 Let f be a GN–polnomial of type (n,t,m,s). The *core* of $V(f)$ is the t-dimensional subspace $\Pi \subset V(f)$ defined by the equations $x_{t+1} = \ldots = x_n = 0$.

We will call a GN–hypersurface of type (n,t,m,s) *general* if the defining data, namely the polynomials $h_i(y_0,\ldots,y_m)$, $i=0,\ldots,t$, the polynomials $\psi_j(x_{t+1},\ldots,x_n)$, $j=0,\ldots,m$ and the constants $a^{(\ell)}_{u,v}$, $\ell=1,\ldots,t-m$, $u=1,\ldots,t-m-1$, $v=0,\ldots,t$, have been chosen generically.

Proposition 3.3 ([1, Proposition 2.11])

Let $V(f)\subset\mathbb{P}^n$ be a GN–hypersurface of type (n,t,m,s) and degree d. Set $\mu=[\frac{d}{s}]$. Then

i) *$V(f)$ has multiplicity $d-\mu$ at the general point of its core Π.*
ii) *The general $(t+1)$-dimensional subspace $\Pi_\xi\subset\mathbb{P}^n$ through Π cuts out on $V(f)$, off Π, a cone of degree μ whose vertex is a m-dimensional subspace $\Gamma_\xi\subset\Pi$.*
iii) *As Π_ξ varies the corresponding subspace Γ_ξ describes the family of tangent spaces to an m-dimensional unirational subvariety $S(f)$ of Π.*
iv) *If $V(f)$ is general and $\mu>n-t-2$ then $V(f)$ is not a cone.*
v) *The general GN–hypersurface is irreducible.*

Definition 3.4 ([5]) A reduced hypersurface $F=V(f)\subset\mathbb{P}^4$ of degree d is said to be a *Franchetta hypersurface* if it is swept out by a one-dimensional family Σ of planes such that:

- all the planes of the family Σ are tangent to a plane rational curve C (of degree $p>1$) lying on F;
- the family Σ and the curve C are such that for a general hyperplane $H=\mathbb{P}^3\subset\mathbb{P}^4$ passing through C, the intersection $H\cap F$, off the linear span of C, is the union of planes of Σ all tangent to the curve C in the same point p_H.

Remark 3.5 Note that by Proposition 3.3 a GN–hypersurface $X=V(f)\subset\mathbb{P}^4$ of type $(4,2,1,s)$ is a Franchetta hypersurface with core the linear span of the curve C. On the contrary Permutti proved in [11] that a Franchetta hypersurface $V(f)\subset\mathbb{P}^4$ is a GN–hypersurface of type $(4,2,1,s)$. In particular (by Proposition 3.1) a Franchetta hypersurface has vanishing Hessian. This fact can be proved directly see also [11] and [1, Proposition 2.18].

4. A geometrical proof of Gordan–Noether's and Franchetta's classification of hypersurfaces in $\mathbb{P}^4$ with vanishing Hessian

In the previous section we saw that the classes of GN-hypersurfaces of type $(4,2,1,s)$ and of Franchetta hypersurfaces coincide. In this section we use the geometrical methods developed in the first section and some other easy facts to provide a short and self-contained proof of Franchetta characterization of hypersurfaces with vanishing Hessian in $\mathbb{P}^4$, [5]. So we will prove in a geometrical way that the hypersurfaces in $\mathbb{P}^4$ with vanishing Hessian are either cones or Franchetta hypersurfaces. A similar result is not known in higher dimension.

First we give a preliminary result describing a geometrical consequence of the vanishing of the hessian of hypersurfaces in $\mathbb{P}^4$, not cones.

Proposition 4.1

Let $X = V(f) \subset \mathbb{P}^4$ be a reduced hypersurface of degree $d \geq 3$, not a cone. If $X = V(f)$ has vanishing Hessian then $Z(f)^* \subset \mathbb{P}^4$ is an irreducible plane rational curve. Equivalently $Z(f)$ is a cone with vertex a line over an irreducible plane rational curve.

Proof. By Corollary 2.6, we can suppose $\dim(Z(f)) = 3$. Thus $Z(f)^* = S_Z^*$, and $Z(f) = V(g)$ with $g \in k[y_0, \dots, y_4]$ an irreducible polynomial. Since X is not a cone, by Remark 1.10 and Proposition 1.17, $1 \leq \dim(Z(f)^*) \leq \dim(\mathrm{Bs}(\psi_g)) \leq 2$.

Assume first $\dim(Z(f)^*) = 2$ so that $Z(f)^*$ is an irreducible component of $\mathrm{Bs}(\psi_g)$. Consider the intersection between the closure of the fibers on two different general points, s_1, $s_2 \in Z(f)^*$. The fiber on each of these points has dimension two, so there exists $t \in \overline{\psi_g^{-1}(s_1)} \cap \overline{\psi_g^{-1}(s_2)}$. By Proposition 1.18, the lines $\langle s_i, t\rangle$, $i = 1, 2$, are contained in $\mathrm{Bs}(\psi_g)$ and hence in the irreducible component of it containing s_1 and s_2. Since s_1 and s_2 are general points in $Z(f)^*$, $Z(f)^*$ is the unique irreducible component of $\mathrm{Bs}(\psi_g)$ containing them. Furthermore $Z(f)^*$ is a ruled surface because through a general point $s \in Z(f)^*$ there passes a line ℓ_s contained in $Z(f)^*$. Moreover $Z(f)^*$ is a cone because $\ell_{s_1} \cap \ell_{s_2} \neq \varnothing$ for $s_1, s_2 \in Z(f)^*$ general points, whence by Remark 2.1, X is a cone.

Thus we can assume $\dim(Z(f)^*) = 1$. Let s_1 and s_2 be two general points of $Z(f)^*$. Then the intersection $\overline{\psi_g^{-1}(s_1)} \cap \overline{\psi_g^{-1}(s_2)}$ is a surface, say R, contained in $\mathrm{Bs}(\psi_g)$ and hence an irreducible component of $\mathrm{Bs}(\psi_g)$ since $\dim(\mathrm{Bs}(\psi_g)) \leq 2$. Note that the surface R does not depend on s_1 and s_2, general points of $Z(f)^*$.

Furthermore for every point $t \in R$ and for a general point $s \in Z(f)^*$, by Proposition 1.18, the line $\langle s, t\rangle$ is contained in $\mathrm{Bs}(\psi_g) \cap \overline{\psi_g^{-1}(s)}$, and hence in R. It follows that $Z(f)^*$ is contained in the vertex of the surface R, and that R (and in fact the intersection of two general fibers of ψ_g) is a plane ($Z(f)^*$ is not a line by assumption, so it cannot be contained in the intersection of two or more planes).

In other words $Z(f)^*$ is a plane curve, whose linear span $\Pi = R$ is an irreducible component of $\mathrm{Bs}(\psi_g)$. Since $S_Z^* = Z(f)^*$, Proposition 1.18 and the same argument used above show that the plane $\Pi = \langle Z(f)^*\rangle$ is an irreducible component of $\mathrm{Sing}(X)$. Note also that $Z(f)^*$ is rational. In fact the map ψ_g is a rational dominant map from $\mathbb{P}^4$ to $Z(f)^*$, so $Z(f)^*$ is a unirational curve and hence a rational curve.

Since $Z(f)^* = S_Z^* \subset \Pi = \mathbb{P}^2$ is an irreducible rational plane curve (not a line), $Z(f)$ is a cone with vertex a line $L = \Pi^* = \mathbb{P}^1$ over an irreducible plane curve Γ (of degree ≥ 2), which is the dual curve of $Z(f)^*$ in the plane Π. Furthermore Γ is a rational curve because the Gauss map of the curve $Z(f)^*$ is birational. □

The description given in Proposition 4.1 is crucial to prove that a projective hypersurface $X = V(f)$ in $\mathbb{P}^4$ with vanishing Hessian which is not a cone is a Francehtta hypersurface. The following result finally gives a characterization of hypersurfaces in $\mathbb{P}^4$ with vanishing Hessian, which are not cones.

Theorem 4.2

Let $X = V(f) \subset \mathbb{P}^4$ be an irreducible and reduced hypersurface of degree $d \geq 3$, not a cone. The following conditions are equivalent:

i) *$X = V(f)$ has vanishing Hessian.*

ii) *$X = V(f)$ is a Franchetta hypersurface.*

iii) *$X^* = V(f)^*$ is a scroll surface of degree d, having a line directrix L of multiplicity e, sitting in a 3-dimensional rational cone $W(f)$ with vertex L, and the general plane ruling of the cone cuts $V(f)^*$ off L along $\mu \leq e$ lines of the scroll, all passing through the same point of L.*

iv) *$X = V(f)$ is a general GN–hypersurface of type $(4, 2, 1, s)$, with $\mu = [\frac{d}{s}]$, which has a plane of multiplicity $d - \mu$.*

In particular, $X^* = V(f)^*$ is smooth if and only if $d = 3$, $X^* = V(f)^*$ is a rational normal scroll of degree 3 and $X = V(f)$ contains a plane, the orthogonal of the line directrix of $X^* = V(f)^*$, with multiplicity 2.

Proof. Note that conditions ii) and iii) are easily seen to be equivalent (the directrix line L of X^* is the dual of the plane which is the linear span of the curve C of the Franchetta hypersurface). We shall provide more details below. By Remark 3.5, the equivalence of ii) and iv) is clear. Condition iv) implies condition i) by Proposition 3.1. Thus to finish the proof it is sufficient to prove that a hypersurface $X = V(f) \subset \mathbb{P}^4$ with vanishing Hessian, not a cone, is a Franchetta hypersurface, or equivalently that it is as in case iii).

By Proposition 4.1, we have that $Z(f)^* \subset \mathrm{Sing}(X) \subset X = V(f)$ is an irreducible plane rational curve, whose linear span $\Pi = \mathbb{P}^2$, is an irreducible component of $\mathrm{Sing}(X)$. Therefore $Z(f) \subset \mathbb{P}^{4*}$ is a cone of vertex the line $L = \Pi^* = \mathbb{P}^1$ over an irreducible plane curve Γ, the dual of $Z(f)^*$ as a plane curve.

Consider now a general hyperplane $H \subset \mathbb{P}^4$ passing through the plane Π. The intersection $X \cap H$ is a hypersurface of degree d in $H = \mathbb{P}^3$ containing the plane Π with a certain multiplicity $m > 0$. Note also that the point $h = H^* \in L = \Pi^*$ (because $\Pi \subset H$), whence $\pi_h(Z(f)) \subset \mathbb{P}^3$ is a non-degenerate surface naturally embedded in the dual space of H. More precisely $\pi_h(Z(f))$ is a cone with vertex the point $p_L = \pi_h(L)$ over the plane curve $\hat{\Gamma} = \pi_h(\Gamma) \simeq \Gamma$.

By Lemma 1.20 we infer that $Z(V(f) \cap H) \subset \pi_h(Z(f)) \subset \mathbb{P}^{3*}$, so that by Proposition 1.4 the hypersurface $V(f) \cap H \subset H = \mathbb{P}^3$ has vanishing Hessian. By Proposition 2.5 it follows that either $V(f) \cap H$ is a cone over a plane curve with vertex a point or $V(f) \cap H$ consists of $d - m$ distinct planes, eventually counted with multiplicity, passing through a line. In the first case $Z(V(f) \cap H)$ would be a plane in $\mathbb{P}^{3*}$, which is impossible because $\pi_h(Z(f))$ is a non degenerate cone with vertex a point.

Therefore $Z(V(f) \cap H)$ is a line l_H in $H^* = \mathbb{P}^3$. The line l_H is contained in $\overline{\phi_f(H)}$ and, by Lemma 1.20, $\overline{\phi_f(H)} \subseteq \Pi_H := \langle h, l_H \rangle$. We now prove that $\overline{\phi_f(H)} = \Pi_H$. Indeed, for a general point q, $\overline{\phi_f^{-1}(\phi_f(q))}$ is a line, we call it L_q. The closure of the fiber of $\phi_{f|H}$ passing through q is either the point q or the line L_q. If it were the point q, $\dim(\overline{\phi_f(H)}) = 3$, which is impossible, because $\overline{\phi_f(H)} \subseteq \Pi_H$. Hence it is the line L_q and $\dim(\overline{\phi_f(H)}) = 2$, i.e. $\overline{\phi_f(H)} = \Pi_H$.

Therefore $V(f) \cap H$, off Π, is a union of $d - m$ planes passing through the line $T = Z(V(f) \cap H)^* \subset H$ (here duality is considered between H and H^*).

Moreover

$$\phi_{V(f)\cap H}(V(f) \cap H) = \{p_1, \ldots, p_{d-m}\} \in l_H$$

and

$$\overline{\phi_f(V(f) \cap H)} = \langle h, p_1 \rangle \cup \ldots \cup \langle h, p_{d-m} \rangle \subset \Pi_H = \overline{\phi_f(H)}.$$

Varying $H \supset \Pi$ we deduce that X^* is a scroll surface, having as line directrix L and such that the general plane ruling of the cone $Z(f)$ cuts X^* off L along $d-m = \mu$ lines of the scroll, all passing through the same point $h \in L$. The scroll surface $X^* \subset \mathbb{P}^{4*}$ is a non-developable surface, in fact $(X^*)^* = X$ is a hypersurface in $\mathbb{P}^4$ (cf. [1, §1.2]). Moreover $\deg(X^*) = \deg((X^*)^*) = \deg(X) = d$ (cf. [1, §1.2]) and $X \subset \mathbb{P}^4$ is a hypersurface as in iii).

This geometrical description also assures that a general H through Π cuts X along $d-m$ distinct planes. For such a general H, let $z = \psi_g(H) \in Z(f)^*$ ($Z(f)^*$ is the plane curve dual of $Z(f)$ with respect to the plane Π). Let Π_H^* be the dual of the plane with respect to the ambient space $\mathbb{P}^4$. Since $\Pi_H^* = T_z(Z(f)^*) = T$, the line of intersection of the planes in $V(f) \cap H$ is the tangent line to the plane curve $Z(f)^*$ at the point z. In conclusion $X = V(f) \subset \mathbb{P}^4$ is a Franchetta hypersurface, where we can take as the one dimensional family Σ of planes contained in X exactly the intersection of a general $\mathbb{P}^3$ through Π with $X = V(f)$ and we consider as the curve C (cf. Definition 3.4) the curve $Z(f)^*$. □

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