On border basis and Gröbner basis schemes

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Abstract

Hilbert schemes of zero-dimensional ideals in a polynomial ring can be covered with suitable affine open subschemes whose construction is achieved using border bases. Moreover, border bases have proved to be an excellent tool for describing zero-dimensional ideals when the coefficients are inexact. And in this situation they show a clear advantage with respect to Gröbner bases which, nevertheless, can also be used in the study of Hilbert schemes, since they provide tools for constructing suitable stratifications.

In this paper we compare Gröbner basis schemes with border basis schemes. It is shown that Gröbner basis schemes and their associated universal families can be viewed as weighted projective schemes. A first consequence of our approach is the proof that *all* the ideals which define a Gröbner basis scheme and are obtained using Buchberger's Algorithm, are equal. Another result is that if the origin (i.e. the point corresponding to the unique monomial ideal) in the Gröbner basis scheme is smooth, then the scheme itself is isomorphic to an affine space. This fact represents a remarkable difference between border basis and Gröbner basis schemes. Since it is natural to look for situations where a Gröbner basis scheme and the corresponding border basis scheme are equal, we address the issue, provide an answer, and exhibit some consequences. Open problems are discussed at the end of the paper.

1. Introduction

This paper has three main sources and one ancestor. Let me be more specific.

Source 1. Given a zero-dimensional ideal I in a polynomial ring, and assuming that the coefficients of the generating polynomials are inexact, what is the best way of describing I? The idea that Gröbner bases are not suitable for computations with inexact data has been brought to light by Stetter (see [18]) and other numerical analysts. Gröbner bases are inadequate due to the rigidity imposed by the term ordering. The class of border bases is more promising. A pioneering paper on border bases is [15] and a detailed description is contained in [12, Section 6.4].

Source 2. The possibility of parametrizing families of schemes with a scheme is a remarkable peculiarity of algebraic geometry. Hilbert schemes are one instance of this phenomenon, and consequently are widely studied. If we let $P = K[x_1, \ldots, x_n]$, Hilbert schemes of zero-dimensional ideals in P can be covered by affine open subschemes which parametrize all the subschemes $\mathrm{Spec}(P/I)$ of the affine space $\mathbb{A}_K^n$ such that P/I has a fixed basis. What is interesting is that the construction of such subschemes is performed using border bases (see for instance [7, 8, 14]).

Source 3. Despite their inability to treat inexact data well, Gröbner bases can nevertheless be used in the study of Hilbert schemes, since with their help it is possible to construct suitable stratifications. Among the vast literature on this subject let me mention the two fairly recent articles [3, 16] and the bibliography quoted therein.

The three main sources are now described; it remains to reveal the ancestor. It is paper [13] where we tried to extend to border bases a very nice property of Gröbner bases, the possibility of connecting every ideal to its leading term ideal via a flat deformation. We were able to get partial results, so that the connectedness of border basis schemes is still an open problem (see Question 2 at the end of this paper).

So what is the content of the next pages? The main idea is to compare Gröbner basis schemes (see Definition 2.4) with border basis schemes. We also define a universal family (see Definition 2.6), and the first main result is Theorem 2.8 where it is shown that Gröbner basis schemes and their associated universal families can be endowed with a graded structure where the indeterminates have positive weights. In other words, they can be viewed as weighted projective schemes (see Remark 2.12). The second main result is Theorem 2.9 where the comparison of the two schemes is fully described. In particular, it is shown that Gröbner basis schemes can be obtained as sections of border basis schemes with suitable linear spaces. Since our description of Gröbner basis schemes is not directly linked to the concept of Gröbner basis, we prove in Corollary 2.11 that indeed our definition is well-placed.

Section 3 is devoted to exhibiting some consequences of the above mentioned results. Let me explain the first one. In the literature Gröbner basis schemes are mostly described using Buchberger's Algorithm. However, this approach has a drawback, since the reduction process in the algorithm is far from being unique, and the consequence is that the description of the Gröbner basis scheme is *a priori* not canonical. A first consequence of our approach is the proof that *all* the ideals obtained using Buchberger's Algorithm are equal (see Proposition 3.6) and coincide with the ideal defined in this paper (see Proposition 3.5).

Another remark is made in Corollary 3.7 where it is shown that if the origin (i.e.

the point corresponding to the unique monomial ideal) in the Gröbner basis scheme is smooth, then the scheme itself is isomorphic to an affine space. This fact represents a remarkable difference between border basis and Gröbner basis schemes (see Example 3.9).

After Theorem 2.9 it is natural to look for situations where a Gröbner basis scheme and the corresponding border basis scheme are the same. The answer is given in Proposition 3.11 and a nice consequence is shown in Corollary 3.13.

Doing mathematics is looking for solutions to problems, a process which inevitably sparks new questions. This paper is no exception; in particular, two open questions are presented at the end of Section 3.

> *Judge others by their questions*
> *rather than by their answers.*
> (François-Marie Arouet (Voltaire))

Unless explicitly stated otherwise, we use definitions and notation introduced in [11, 12, 13]. All the experimental computation was done with the computer algebra system CoCoA (see [2]).

2. Border basis and Gröbner basis schemes

In the following we let K be a field, $P = K[x_1, \ldots, x_n]$ a polynomial ring, and $I \subset P$ a zero-dimensional ideal. Recall that an *order ideal* $\mathcal{O}$ is a finite set of terms in $\mathbb{T}^n = \mathbb{T}(x_1, \ldots, x_n) = \{x_1^{\alpha_1} \cdots x_n^{\alpha_n} \mid \alpha_i \geq 0\}$ such that all divisors of a term in $\mathcal{O}$ are also contained in $\mathcal{O}$. The set $\partial \mathcal{O} = (x_1 \mathcal{O} \cup \cdots \cup x_n \mathcal{O}) \setminus \mathcal{O}$ is called the *border* of $\mathcal{O}$.

DEFINITION 2.1   Let $\mathcal{O} = \{t_1, \ldots, t_\mu\}$ be an order ideal and $\partial \mathcal{O} = \{b_1, \ldots, b_\nu\}$ its border.

 a) A set of polynomials $\{g_1, \ldots, g_\nu\} \subseteq I$ is called an $\mathcal{O}$-*border prebasis* of I if it is of the form $g_j = b_j - \sum_{i=1}^{\mu} a_{ij} t_i$ with $a_{ij} \in K$.
 b) An $\mathcal{O}$-border prebasis of I is called an $\mathcal{O}$-*border basis* of I if $P = I \oplus \langle \mathcal{O} \rangle_K$.

It is known that if I has an $\mathcal{O}$-border basis, then such $\mathcal{O}$-border basis of I is unique (see [12, Proposition 6.4.17]).

Proposition 2.2 (Border bases and multiplication matrices)
   Let $\mathcal{O} = \{t_1, \ldots, t_\mu\}$ *be an order ideal of monomials, let the set* $\{g_1, \ldots, g_\nu\}$ *be an* $\mathcal{O}$*-border prebasis, and let* I *be the ideal generated by* $\{g_1, \ldots, g_\nu\}$. *Then the following conditions are equivalent*

 a) *The set* $\{g_1, \ldots, g_\nu\}$ *is the* $\mathcal{O}$*-border basis of* I.
 b) *The formal multiplication matrices of* $\{g_1, \ldots, g_\nu\}$ *are pairwise commuting.*

Proof. See [12, Definition 6.4.29] and Theorem 6.4.30.   □

DEFINITION 2.3   Let $\{c_{ij} \mid 1 \leq i \leq \mu,\ 1 \leq j \leq \nu\}$ be a set of new indeterminates.

a) The *generic $\mathcal{O}$-border prebasis* is the set of polynomials $G = \{g_1, \ldots, g_\nu\}$ in $K[x_1, \ldots, x_n, c_{11}, \ldots, c_{\mu\nu}]$ given by

$$g_j = b_j - \sum_{i=1}^{\mu} c_{ij} t_i.$$

b) For $k = 1, \ldots, n$, let $\mathcal{A}_k \in \mathrm{Mat}_\mu(K[c_{ij}])$ be the kth formal multiplication matrix associated to G (cf. [12, Definition 6.4.29]). It is also called the kth *generic multiplication matrix* with respect to $\mathcal{O}$.

c) The ideal of $K[c_{11}, \ldots, c_{\mu\nu}]$ generated by the entries of $\mathcal{A}_k \mathcal{A}_\ell - \mathcal{A}_\ell \mathcal{A}_k$ with $1 \leq k < \ell \leq n$ defines an affine subscheme of $\mathbb{A}^{\mu\nu}$ which will be denoted by $\mathbb{B}_\mathcal{O}$ and called the *$\mathcal{O}$-border basis scheme*. Its defining ideal will be denoted by $I(\mathbb{B}_\mathcal{O})$, and its coordinate ring $K[c_{11}, \ldots, c_{\mu\nu}]/I(\mathbb{B}_\mathcal{O})$ will be denoted by $B_\mathcal{O}$.

The reason why it is called the $\mathcal{O}$-border basis scheme is the following. When we apply the substitution $\Sigma(c_{ij}) = \alpha_{ij}$ to G, a point $(\alpha_{ij}) \in K^{\mu\nu}$ yields a border basis if and only if $\Sigma(\mathcal{A}_k) \Sigma(\mathcal{A}_\ell) = \Sigma(\mathcal{A}_\ell) \sigma(\mathcal{A}_k)$ for $1 \leq k < \ell \leq n$ (see Proposition 2.2). Thus the K-rational points of $\mathbb{B}_\mathcal{O}$ are in 1–1 correspondence with the $\mathcal{O}$-border bases of zero-dimensional ideals in P, and therefore are in 1–1 correspondence with *all zero-dimensional ideals I in P such that $\overline{\mathcal{O}}$ is a basis of P/I as a K-vector space.*

Next, we are going to define $(\mathcal{O}, \sigma)$-Gröbner basis schemes, and to do this an extra bit of notation is required. Let $\mathcal{O} = \{t_1, \ldots, t_\mu\}$ be an order ideal. Then the set of minimal generators of the monoideal $\mathbb{T}^n \setminus \mathcal{O}$ (also called the *corners* of $\mathcal{O}$) is denoted by $c\mathcal{O}$, and we denote by η the cardinality of $c\mathcal{O}$. Since $c\mathcal{O} \subseteq \partial\mathcal{O}$, it follows that $\eta \leq \nu$, and we label the elements in $\partial\mathcal{O}$ so that $c\mathcal{O} = \{b_1, \ldots, b_\eta\}$.

We let σ be a term ordering on $\mathbb{T}^n$ and recall that if I is an ideal in the polynomial ring P, we denote the order ideal $\mathbb{T}^n \setminus \mathrm{LT}_\sigma(I)$ by $\mathcal{O}_\sigma(I)$. Moreover, we denote by $S_{\mathcal{O},\sigma}$ the set $\{c_{ij} \in \{c_{11}, \ldots, c_{\mu\nu}\} \mid b_j >_\sigma t_i\}$, by $L_{\mathcal{O},\sigma}$ the ideal generated by $\{c_{11}, \ldots, c_{\mu\nu}\} \setminus S_{\mathcal{O},\sigma}$ in $K[c_{11}, \ldots, c_{\mu\nu}]$, by $S_{c\mathcal{O},\sigma}$ the intersection $S_{\mathcal{O},\sigma} \cap \{c_{11}, \ldots, c_{\mu\eta}\}$, and by $L_{c\mathcal{O},\sigma}$ the ideal generated by $\{c_{11}, \ldots, c_{\mu\eta}\} \setminus S_{c\mathcal{O},\sigma}$ in $K[c_{11}, \ldots, c_{\mu\eta}]$. Furthermore we denote the cardinality of $S_{c\mathcal{O},\sigma}$ by $s(c\mathcal{O}, \sigma)$.

DEFINITION 2.4   For $j = 1, \ldots, \nu$ we define g_j^* in the following way.

$$g_j^* = b_j - \sum_{\{i \mid b_j >_\sigma t_i\}} c_{ij} t_i = b_j - \sum_{c_{ij} \in S_{\mathcal{O},\sigma} \cap \{c_{1j}, \ldots, c_{\mu j}\}} c_{ij} t_i.$$

a) The *generic $(\mathcal{O}, \sigma)$-Gröbner prebasis* is the set of polynomials $\{g_1^*, \ldots, g_\eta^*\}$.

b) The ideal $(L_{\mathcal{O},\sigma} + I(\mathbb{B}_\mathcal{O})) \cap K[S_{c\mathcal{O},\sigma}]$ of $K[S_{c\mathcal{O},\sigma}]$ defines an affine subscheme of $\mathbb{A}^{s(c\mathcal{O},\sigma)}$ which will be denoted by $\mathbb{G}_{\mathcal{O},\sigma}$ and called the$(\mathcal{O}, \sigma)$-*Gröbner basis scheme*. The defining ideal $(L_{\mathcal{O},\sigma} + I(\mathbb{B}_\mathcal{O})) \cap K[S_{c\mathcal{O},\sigma}]$ will be denoted by $I(\mathbb{G}_{\mathcal{O},\sigma})$ and the coordinate ring $K[S_{c\mathcal{O},\sigma}]/I(\mathbb{G}_{\mathcal{O},\sigma})$ will be denoted by $G_{\mathcal{O},\sigma}$.

We observe that g_j^* is obtained from g_j by setting to zero all the indeterminates in $L_{\mathcal{O},\sigma} \cap \{c_{1j}, \ldots, c_{\mu j}\}$.

EXAMPLE 2.5   We examine the inclusion $c\mathcal{O} \subseteq \partial\mathcal{O}$. If $\mathcal{O} = \{1, x, y, xy\}$ then $c\mathcal{O} = \{x^2, y^2\}$ while $\partial\mathcal{O} = \{x^2, y^2, x^2y, xy^2\}$, so that $c\mathcal{O} \subset \partial\mathcal{O}$. On the other hand, if $\mathcal{O} = \{1, x, y\}$ then $c\mathcal{O} = \partial\mathcal{O} = \{x^2, xy, y^2\}$.

Returning to $\mathcal{O} = \{1, x, y, xy\}$ we observe that $t_1 = 1$, $t_2 = x$, $t_3 = y$, $t_4 = xy$, $b_1 = x^2$, $b_2 = y^2$, $b_3 = x^2y$, $b_4 = xy^2$. Let $\sigma = \mathtt{DegRevLex}$, so that $x >_\sigma y$. Then $L_{\mathcal{O},\sigma} = L_{c\mathcal{O},\sigma} = (c_{42})$, $g_1^* = g_1$, $g_2^* = y^2 - (c_{12} + c_{22}x + c_{32}y)$, $g_3^* = g_3$, $g_4^* = g_4$.

Having introduced the Gröbner basis scheme, we define a naturally associated universal family. To this end we recall the following definition taken from [13] and extend it.

DEFINITION 2.6   The ring $K[x_1, \ldots, x_n, c_{11}, \ldots, c_{\mu\nu}]/(I(\mathbb{B}_{\mathcal{O}}) + (g_1, \ldots, g_\nu))$ will be denoted by $U_{\mathcal{O}}$. The ring $K[x_1, \ldots, x_n, S_{c\mathcal{O},\sigma}]/(I(\mathbb{G}_{\mathcal{O},\sigma}) + (g_1^*, \ldots, g_\eta^*))$ will be denoted by $U_{\mathcal{O},\sigma}$.

  a) The natural homomorphism of K-algebras $\Phi : B_{\mathcal{O}} \longrightarrow U_{\mathcal{O}}$ is called the *universal $\mathcal{O}$-border basis family.*
  b) The natural homomorphism of K-algebras $\Psi : G_{\mathcal{O},\sigma} \longrightarrow U_{\mathcal{O},\sigma}$ is called the *universal $(\mathcal{O}, \sigma)$-Gröbner basis family.*
  c) The induced homomorphism of K-algebras $B_{\mathcal{O}}/\overline{L}_{\mathcal{O},\sigma} \longrightarrow U_{\mathcal{O}}/\overline{L}_{\mathcal{O},\sigma}$ will be denoted by $\overline{\Phi}$.

Remark 2.7   It is known (see [1], and [4, Exercise 15.12, p. 370]) that given power products $t, t_1, \ldots, t_r \in \mathbb{T}^n$ and a term ordering σ such that $t >_\sigma t_i$ for $i = 1, \ldots r$, then there exists a system V of positive weights on $x_1, \ldots, x_n$ (i.e a matrix $V \in \mathrm{Mat}_{1,n}(\mathbb{N}_+)$) such that $\deg_V(t) > \deg_V(t_i)$ for $i = 1, \ldots, r$.

We are ready to prove an important property of some ideals described before. To help the reader, we observe that for simplicity we write $\mathbf{x}$ for $x_1, \ldots, x_n$ and $\mathbf{c}$ for $c_{11}, \ldots c_{\mu\nu}$.

Theorem 2.8

There exist a system W of positive weights on the elements of $S_{c\mathcal{O},\sigma}$, a system $\overline{W}$ of positive weights on the elements of $S_{\mathcal{O},\sigma}$, and a system V of positive weights on $\mathbf{x}$ such that the following conditions hold true.

  a) *The system $\overline{W}$ is an extension of the system W.*
  b) *The ideal $I(\mathbb{G}_{\mathcal{O},\sigma})$ in $K[S_{c\mathcal{O},\sigma}]$ is W-homogeneous.*
  c) *The ideal $I(\mathbb{G}_{\mathcal{O},\sigma}) + (g_1^*, \ldots, g_\eta^*)$ in $K[\mathbf{x}, S_{c\mathcal{O},\sigma}]$ is (V, W)-homogeneous.*
  d) *The image of $I(\mathbb{B}_{\mathcal{O}})$ in $K[S_{\mathcal{O},\sigma}] \cong K[\mathbf{c}]/L_{\mathcal{O},\sigma}$ is $\overline{W}$-homogeneous.*
  e) *The image of $I(\mathbb{B}_{\mathcal{O}}) + (g_1^*, \ldots, g_\nu^*)$ in $K[\mathbf{x}, S_{\mathcal{O},\sigma}] \cong K[\mathbf{x}, \mathbf{c}]/L_{\mathcal{O},\sigma}$ is $(V, \overline{W})$-homogeneous.*

Proof. The definition of $S_{c\mathcal{O},\sigma}$ and Remark 2.7 imply that there exists a system V of positive weights on $\mathbf{x}$ such that $\deg_V(b_j) > \deg_V(t_i)$ for every $j = 1, \ldots, \eta$ and every $t_i \in \mathrm{Supp}(g_j^* - b_j)$. We define W by giving the c_{ij}'s suitable positive weights, so that all elements g_j^* in the generic $(\mathcal{O}, \sigma)$-Gröbner prebasis are (V, W)-homogeneous when they are viewed as polynomials in $K[\mathbf{x}, S_{c\mathcal{O},\sigma}]$.

Then we choose a $\deg_{(V,W)}$-compatible term ordering $\overline{\sigma}$ on $\mathbb{T}(\mathbf{x}, S_{c\mathcal{O},\sigma})$ with the property that for every $t, t' \in \mathbb{T}(S_{c\mathcal{O},\sigma})$, $x_1^{a_1} \cdots x_n^{a_n} t >_{\overline{\sigma}} x_1^{b_1} \cdots x_n^{b_n} t'$ if they have the same (V,W)-degree and $x_1^{a_1} \cdots x_n^{a_n} >_{\sigma} x_1^{b_1} \cdots x_n^{b_n}$. If we use the $\overline{\sigma}$-division algorithm with respect to the tuple $(g_1^*, \ldots, g_\eta^*)$, we can express every element $b_j \in \partial\mathcal{O} \setminus c\mathcal{O}$ as a linear combination of those elements in $\mathcal{O}$ which are σ-smaller than b_j. Since all the g_i^* are monic and homogeneous, the coefficients h_{ij} of these linear combinations are homogeneous polynomials in the c_{ij}'s. We define $\overline{W}$ by putting $\deg_{\overline{W}}(c_{ij}) = \deg_W(h_{ij})$ for $c_{ij} \in S_{\mathcal{O},\sigma} \cap \{c_{1j}, \ldots, c_{\mu j}\}$ and $j = \eta + 1, \ldots, \nu$. We observe that $\overline{W}$ does not depend on the choice of the order in the division algorithm, it only depends on $\mathcal{O}$, σ, V. At this point we have proved statement $a)$ and have shown that the polynomials $g_1^*, \ldots, g_\nu^*$ are $(V,\overline{W})$-homogeneous which implies that $d)$ and $e)$ are equivalent. Moreover, we observe that $b)$ follows from $d)$, while $c)$ and $d)$ follow from $e)$, so we only need to prove $d)$. Multiplication by x_i yields a graded homomorphism between $(V,\overline{W})$-graded free $K[\mathbf{x}, \mathbf{c}]/L_{\mathcal{O},\sigma}$-modules, therefore the multiplication matrices are homogeneous (see [12, Definition 4.7.1] and Proposition 4.7.4). Consequently, the image of the ideal $I(\mathbb{B}_\mathcal{O})$ modulo $L_{\mathcal{O},\sigma}$ is $\overline{W}$-homogeneous and the proof is complete.   □

In the sequel we consider the following commutative diagram of canonical homomorphisms

$$
\begin{array}{ccc}
G_{\mathcal{O},\sigma} & \xrightarrow{\varphi} & B_\mathcal{O}/\overline{L}_{\mathcal{O},\sigma} \\[1em]
\Big\downarrow{\scriptstyle\Psi} & & \Big\downarrow{\scriptstyle\overline{\Phi}} \\[1em]
U_{\mathcal{O},\sigma} & \xrightarrow{\vartheta} & U_\mathcal{O}/\overline{L}_{\mathcal{O},\sigma}
\end{array}
\tag{1}
$$

i.e.

$$
\begin{array}{ccc}
K[S_{c\mathcal{O},\sigma}]/I(\mathbb{G}_{\mathcal{O},\sigma}) & \xrightarrow{\varphi} & K[\mathbf{c}]/(L_{\mathcal{O},\sigma} + I(\mathbb{B}_\mathcal{O})) \\[1em]
\Big\downarrow{\scriptstyle\Psi} & & \Big\downarrow{\scriptstyle\overline{\Phi}} \\[1em]
K[\mathbf{x}, S_{c\mathcal{O},\sigma}]/(I(\mathbb{G}_{\mathcal{O},\sigma}) + (g_1^*, \ldots, g_\eta^*)) & \xrightarrow{\vartheta} & K[\mathbf{x}, \mathbf{c}]/(L_{\mathcal{O},\sigma} + I(\mathbb{B}_\mathcal{O}) + (g_1, \ldots g_\nu)).
\end{array}
$$

We recall the equality $I(\mathbb{G}_\mathcal{O}) = (L_{\mathcal{O},\sigma} + I(\mathbb{B}_\mathcal{O})) \cap K[S_{c\mathcal{O},\sigma}]$ from which the homomorphism φ derives. The homomorphism ϑ is obtained as follows: let $\Theta : K[\mathbf{x}, S_{c\mathcal{O},\sigma}] \longrightarrow K[\mathbf{x}, \mathbf{c}]$ be the natural inclusion of polynomial rings. Then clearly $I(\mathbb{G}_{\mathcal{O},\sigma}) + (g_1^*, \ldots, g_\eta^*) \subseteq \Theta^{-1}(L_{\mathcal{O},\sigma} + I(\mathbb{B}_\mathcal{O}) + (g_1, \ldots g_\nu))$.

We are ready to state the main result of this section. To prove it we are going to make extensive use of the above diagram (1).

Theorem 2.9 (Gröbner and border)

Let $\mathcal{O} = \{t_1, \ldots, t_\mu\}$ be an order ideal of monomials in P and let σ be a term ordering on $\mathbb{T}^n$.

a) *The classes of the elements in $\mathcal{O}$ form a $B_{\mathcal{O}}/\overline{L}_{\mathcal{O},\sigma}$-module basis of $U_{\mathcal{O}}/\overline{L}_{\mathcal{O},\sigma}$.*
b) *The classes of the elements in $\mathcal{O}$ form a $G_{\mathcal{O},\sigma}$-module basis of $U_{\mathcal{O},\sigma}$.*
c) *We have the equality $I(\mathbb{G}_{\mathcal{O},\sigma}) + (g_1^*, \ldots, g_\eta^*) = \vartheta^{-1}(L_{\mathcal{O},\sigma} + I(\mathbb{B}_{\mathcal{O}}) + (g_1, \ldots g_\nu))$.*
d) *The maps φ and ϑ in the above diagram are isomorphisms.*

Proof. We observe that φ is injective by definition. The fact that $\Phi : B_{\mathcal{O}} \longrightarrow U_{\mathcal{O}}$ is free with basis $\overline{\mathcal{O}}$ and injective is proved in [13, Theorem 3.4.] Passing to the quotient modulo $\overline{L}_{\mathcal{O},\sigma}$, we deduce that $\overline{\Phi} : B_{\mathcal{O}}/\overline{L}_{\mathcal{O},\sigma} :\longrightarrow U_{\mathcal{O}}/\overline{L}_{\mathcal{O},\sigma}$ is free with basis $\overline{\mathcal{O}}$ and injective, so that a) is proved. We divide the proof of b) into two claims.

Claim 1. $\overline{\mathcal{O}}$ generates. According to Theorem 2.8, we may choose positive weights W on the elements of $S_{c\mathcal{O},\sigma}$ and positive weights V on $\mathbf{x}$ so that $I(\mathbb{G}_{\mathcal{O},\sigma})$ is a W-homogeneous ideal of $K[S_{c\mathcal{O},\sigma}]$ and $I(\mathbb{G}_{\mathcal{O},\sigma}) + (g_1^*, \ldots, g_\eta^*)$ is a (V,W)-homogeneous ideal of $K[\mathbf{x}, S_{c\mathcal{O},\sigma}]$. Following the lines of the proof of Theorem 2.8, we choose a $\deg_{(U,W)}$-compatible term ordering $\overline{\sigma}$ on $\mathbb{T}(\mathbf{x}, S_{c\mathcal{O},\sigma})$ with the property that for every $t, t' \in \mathbb{T}(S_{c\mathcal{O},\sigma})$, $x_1^{a_1} \cdots x_n^{a_n} t >_{\overline{\sigma}} x_1^{b_1} \cdots x_n^{b_n} t'$ if they have the same (V,W)-degree and $x_1^{a_1} \cdots x_n^{a_n} >_\sigma x_1^{b_1} \cdots x_n^{b_n}$. We observe that $\mathcal{O}$ is the complement in $\mathbb{T}^n$ of the monoideal generated by $c\mathcal{O}$, hence if we use the $\overline{\sigma}$-division algorithm with respect to the tuple $(g_1^*, \ldots, g_\eta^*)$, we can express every polynomial in $K[\mathbf{x}, S_{c\mathcal{O},\sigma}]$ as a linear combination of elements in $\mathcal{O}$, modulo $(g_1^*, \ldots, g_\eta^*)$.

Claim 2. $\overline{\mathcal{O}}$ is linearly independent over $G_{\mathcal{O},\sigma}$. Let $f = \sum_{i=1}^\mu f_i t_i \in K[\mathbf{x}, S_{c\mathcal{O},\sigma}]$ and assume that $f = 0$ modulo $(I(\mathbb{G}_{\mathcal{O},\sigma}) + (g_1^*, \ldots, g_\eta^*))$. The map ϑ sends $\overline{f}$ to zero, hence we have $\sum_{i=1}^\mu f_i t_i = 0$ modulo $(L_{\mathcal{O},\sigma} + I(\mathbb{B}_{\mathcal{O}}) + (g_1^*, \ldots, g_\nu^*))$. By what we have proved before, $\overline{\Phi}$ is free with basis $\overline{\mathcal{O}}$, hence we deduce that $f_i \in L_{\mathcal{O},\sigma} + I(\mathbb{B}_{\mathcal{O}})$, hence $f_i \in (L_{\mathcal{O},\sigma} + I(\mathbb{B}_{\mathcal{O}})) \cap K[S_{c\mathcal{O},\sigma}]$ for $i = 1, \ldots \mu$. The equality $(L_{\mathcal{O},\sigma} + I(\mathbb{B}_{\mathcal{O}})) \cap K[S_{c\mathcal{O},\sigma}] = I(\mathbb{G}_{\mathcal{O},\sigma})$ yields the conclusion and the proof of b) is complete.

The proof of c) uses the same argument as above, which shows that if $\vartheta(\overline{f}) = 0$ then $\overline{f} = 0$. Finally we prove d). At this point we know that diagram (1) is commutative, all the homomorphisms are injective, and both Ψ and $\overline{\Phi}$ are free with basis $\overline{\mathcal{O}}$. Due to this particular structure, the surjectivity of ϑ is equivalent to the surjectivity of φ. Since all the indeterminates which generate $L_{\mathcal{O},\sigma}$ are killed, we need to show that all the indeterminates in $S_{\mathcal{O},\sigma}$ can be expressed as polynomial functions of the indeterminates in $S_{c\mathcal{O},\sigma}$. We consider the generic $(\mathcal{O},\sigma)$-Gröbner prebasis $\{g_1^*, \ldots, g_\eta^*\}$ and argue as in the proof of Proposition 2.8. For every $j = \eta + 1, \ldots, \nu$ we produce elements $b_j - \sum_{\{i \mid b_j >_\sigma t_i\}} h_{ij} t_i$ which are in the ideal $(g_1^*, \ldots, g_\eta^*) \subseteq (g_1^*, \ldots, g_\nu^*)$. Consequently, modulo $(L_{\mathcal{O},\sigma} + I(\mathbb{B}_{\mathcal{O}}) + (g_1^*, \ldots, g_\nu^*))$ we have $b_j - \sum_{\{i \mid b_j >_\sigma t_i\}} h_{ij} t_i = 0$ as well as $b_j - \sum_{\{i \mid b_j >_\sigma t_i\}} c_{ij} t_i = 0$ for every $j = \eta + 1, \ldots, \nu$. We deduce the relations $\sum_{\{i \mid b_j >_\sigma t_i\}} (c_{ij} - h_{ij}) t_i = 0$ in $U_{\mathcal{O}}/\overline{L}_{\mathcal{O},\sigma}$ for every $j = \eta + 1, \ldots, \nu$. Using a) we get the relations $c_{ij} = h_{ij}$ in the ring $B_{\mathcal{O}}/\overline{L}_{\mathcal{O},\sigma}$, for every $c_{ij} \in S_{\mathcal{O},\sigma} \setminus S_{c\mathcal{O},\sigma}$, and every $j = \eta + 1, \ldots, \nu$, and the proof is complete. $\square$

Remark 2.10 After the theorem, diagram (1) can be rewritten in the following way.

$$G_{\mathcal{O},\sigma} \overset{\varphi}{\simeq} B_{\mathcal{O}}/\overline{L}_{\mathcal{O},\sigma}$$

$$\downarrow \Psi \qquad\qquad \downarrow \overline{\Phi} \qquad\qquad\qquad (2)$$

$$U_{\mathcal{O},\sigma} \overset{\vartheta}{\simeq} U_{\mathcal{O}}/\overline{L}_{\mathcal{O},\sigma}$$

Corollary 2.11

Let $\mathcal{O} = \{t_1, \ldots, t_\mu\}$ be an order ideal of monomials in P and let σ be a term ordering on $\mathbb{T}^n$.

a) The affine scheme $\mathbb{G}_{\mathcal{O},\sigma}$ parametrizes all zero-dimensional ideals I in P for which $\mathcal{O} = \mathcal{O}_\sigma(I)$.

b) The fibers over the K-rational points of the universal $(\mathcal{O}, \sigma)$-Gröbner family $\Psi : G_{\mathcal{O},\sigma} \longrightarrow U_{\mathcal{O},\sigma}$ are the quotient rings P/I for which I is a zero-dimensional ideal with the property that $\mathcal{O} = \mathcal{O}_\sigma(I)$. Moreover, the reduced σ-Gröbner basis of I is obtained by specializing the $(\mathcal{O}, \sigma)$-Gröbner prebasis $\{g_1^*, \ldots, g_\eta^*\}$ to the corresponding maximal linear ideal.

Proof. The freeness of Ψ implies that a) follows from b), and we prove b) in two steps. A K-rational point of the universal $(\mathcal{O}, \sigma)$-Gröbner basis family can be viewed as the Ψ-fiber over a maximal linear ideal of $G_{\mathcal{O},\sigma}$. The latter is the canonical projection of a maximal linear ideal $\mathfrak{n} = (c_{ij} - a_{ij} \mid c_{ij} \in S_{c\mathcal{O},\sigma}, \ a_{ij} \in K)$ of $K[S_{c\mathcal{O},\sigma}]$. Let us put $IndL = \{(i,j) \mid c_{ij}$ is a generator of $L_{\mathcal{O},\sigma}\}$. The theorem implies that then $\mathfrak{n}$ is the contraction to $K[S_{c\mathcal{O},\sigma}]$ of a maximal linear ideal

$$\mathfrak{m} = (c_{ij} - a_{ij} \mid c_{ij} \in \{c_{11}, \ldots, c_{\mu\nu}\}, \ a_{ij} \in K, \ a_{ij} = 0 \text{ for all } (i,j) \in IndL)$$

of $K[c_{11}, \ldots, c_{\mu\nu}]$. The ideal $\mathfrak{m}$ contains $I(\mathbb{B}_{\mathcal{O}})$, hence if we substitute c_{ij} with a_{ij} in the polynomials $g_1^*, \ldots, g_\nu^*$, we get polynomials $\overline{g}_1, \ldots, \overline{g}_\nu$ in P which form the $\mathcal{O}$-border basis of the ideal $I = (\overline{g}_1, \ldots, \overline{g}_\nu)$. Moreover, by construction we have $\mathrm{LT}_\sigma(\overline{g}_j) = b_j$ for $j = 1, \ldots, \nu$. Hence $\{\overline{g}_1, \ldots, \overline{g}_\eta\}$ is the reduced σ-Gröbner basis of I by [12, Proposition 6.4.18].

Conversely, let I be a zero-dimensional ideal in P such that $\mathcal{O}_\sigma(I) = \mathcal{O}$ and let $\{g_1, \ldots, g_\eta\}$ be its reduced σ-Gröbner basis. Using the division algorithm, we represent all the elements in $\partial\mathcal{O} \setminus c\mathcal{O}$ uniquely (modulo I) as linear combinations of elements in $\mathcal{O}$. In this way, the $\mathcal{O}$-border basis $(g_1, \ldots, g_\nu)$ of I is constructed. Collecting the coefficients, we produce a maximal linear ideal in $B_{\mathcal{O}}$, equivalently a rational point $\mathbf{p}$ of $\mathbb{B}_{\mathcal{O}}$. By construction, $b_j = \mathrm{LT}_\sigma(g_j)$ for $j = 1, ..., \nu$, and hence the coordinates of $\mathbf{p}$ corresponding to the indices ij such that $(i,j) \in IndL$ have to be zero. In conclusion, the point $\mathbf{p}$ corresponds to a maximal linear ideal $\mathfrak{m}$ of $B_{\mathcal{O}}/\overline{L}_{\mathcal{O},\sigma}$ hence to a maximal linear ideal of $G_{\mathcal{O},\sigma}$ by the theorem, hence to a rational point $\mathbf{q}$ of $\mathbb{G}_{\mathcal{O},\sigma}$. The ideal itself is represented via its reduced σ-Gröbner basis $\{\overline{g}_1, \ldots, \overline{g}_\eta\}$ in the Ψ-fiber over $\mathfrak{m}$.

Remark 2.12 Diagram (2) gives rise to the corresponding diagram

$$\begin{array}{ccc}
\mathbb{G}_{\mathcal{O},\sigma} & \cong & \mathrm{Spec}(B_{\mathcal{O}}/\overline{L}_{\mathcal{O},\sigma}) \\
\\
\uparrow\pi_{\Psi} & & \uparrow\pi_{\overline{\Phi}} \\
\\
\mathrm{Spec}(U_{\mathcal{O},\sigma}) & \cong & \mathrm{Spec}(U_{\mathcal{O}}/\overline{L}_{\mathcal{O},\sigma})
\end{array}$$

of affine schemes, but more can be said. Let W, $\overline{W}$, V be systems of positive weights, chosen suitably to satisfy Theorem 2.8. Then $G_{\mathcal{O},\sigma}$ is a W-graded ring, $B_{\mathcal{O}}$ is a $\overline{W}$-graded ring, $U_{\mathcal{O},\sigma}$ is a (V,W)-graded ring, and $U_{\mathcal{O}}/\overline{L}_{\mathcal{O},\sigma}$ is a $(V,\overline{W})$-graded ring.

With the above assumptions we see that diagram (2) gives rise to a diagram

$$\begin{array}{ccc}
\mathrm{Proj}(G_{\mathcal{O},\sigma}) & \cong & \mathrm{Proj}(B_{\mathcal{O}}/\overline{L}_{\mathcal{O},\sigma}) \\
\\
\uparrow\Pi_{\Psi} & & \uparrow\Pi_{\overline{\Phi}} & \quad\quad (3) \\
\\
\mathrm{Proj}(U_{\mathcal{O},\sigma}) & \cong & \mathrm{Proj}(U_{\mathcal{O}}/\overline{L}_{\mathcal{O},\sigma})
\end{array}$$

of projective schemes $\mathrm{Proj}(G_{\mathcal{O},\sigma})$, $\mathrm{Proj}(B_{\mathcal{O}}/\overline{L}_{\mathcal{O},\sigma})$, $\mathrm{Proj}(U_{\mathcal{O},\sigma})$, $\mathrm{Proj}(U_{\mathcal{O}}/\overline{L}_{\mathcal{O},\sigma})$ such that $\mathrm{Proj}(G_{\mathcal{O},\sigma}) \subset \mathbb{P}(W)$, $\mathrm{Proj}(B_{\mathcal{O}}/\overline{L}_{\mathcal{O},\sigma}) \subset \mathbb{P}(\overline{W})$, $\mathrm{Proj}(U_{\mathcal{O},\sigma}) \subset \mathbb{P}(V,W)$, and $\mathrm{Proj}(U_{\mathcal{O}}/\overline{L}_{\mathcal{O},\sigma}) \subset \mathbb{P}(V,\overline{W})$ where $\mathbb{P}(W)$, $\mathbb{P}(\overline{W})$, $\mathbb{P}(V,W)$, and $\mathbb{P}(V,\overline{W})$ are the corresponding weighted projective spaces.

Moreover, let $\mathbf{p} = (a_{ij}) \in \mathbb{G}_{\mathcal{O},\sigma}$ be a rational point, let $I \subset P$ be the corresponding ideal according to Corollary 2.11, let $v_i = \deg(x_i)$ in the V-grading, and let $w_{ij} = \deg(c_{ij})$ in the W-grading. Then it is well-known that the substitution $a_{ij} \longrightarrow t^{w_{ij}}a_{ij}$ gives rise to a flat family of ideals whose general fibers are ideals isomorphic to I, and whose special fiber is the monomial ideal $\mathrm{LT}_\sigma(I)$. In the setting of diagram (2), the rational monomial curve which parametrizes such family is a curve in $\mathbb{G}_{\mathcal{O},\sigma}$ which connects the two points representing I and $\mathrm{LT}_\sigma(I)$. In the setting of diagram (3), the rational monomial curve is simply a point in $\mathrm{Proj}(G_{\mathcal{O},\sigma}) \subset \mathbb{P}(W)$, which represents all the above ideals except the special one.

3. Consequences and problems

We open the section by discussing the relation between our construction of $I(\mathbb{G}_{\mathcal{O}})$ and other constructions described in the literature (see for instance [3, 16]). If one starts with the generic σ-Gröbner prebasis $\{g_1^*, \ldots, g_\eta^*\}$ one can construct an affine subscheme of $\mathbb{A}^{s(c\mathcal{O},\sigma)}$ in the following way. Using Buchberger Algorithm one reduces the critical pairs of the leading terms of the σ-Gröbner prebasis as much as possible. The reduction stops when a polynomial is obtained which is a linear combination of the elements in $\mathcal{O}$ with coefficients in $K[S_{c\mathcal{O},\sigma}]$. Collecting all coefficients obtained in this way for all the critical pairs, one gets a set which generates an ideal J in $K[S_{c\mathcal{O},\sigma}]$. Clearly each zero of J gives rise to a specialization of the generic σ-Gröbner prebasis which is, by construction, the reduced σ-Gröbner basis of a zero-dimensional ideal I in P for which

$\mathcal{O} = \mathcal{O}_\sigma(I)$. However, there is a drawback; the reduction procedure in Buchberger Algorithm is far from being unique. This observation leads to the following definition which puts the above description in a more formal context.

DEFINITION 3.1   Let J be an ideal in $K[S_{c\mathcal{O},\sigma}]$ such that $\mathrm{Spec}(K[S_{c\mathcal{O},\sigma}]/J)$ parametrizes all zero-dimensional ideals I in P for which $\mathcal{O} = \mathcal{O}_\sigma(I)$. Then J is called an $(\mathcal{O}, \sigma)$-*parametrizing ideal*.

Let J be an an $(\mathcal{O}, \sigma)$-parametrizing ideal and assume that there exists a finite set $\mathcal{S}$ of polynomials of type $\sum_{j=1}^\eta f_j g_j^* = \sum_{t_i \in \mathcal{O}} r_i t_i$ where the f_i's are polynomials in $K[\mathbf{x}, S_{c\mathcal{O},\sigma}]$, the r_i's are polynomials in $K[S_{c\mathcal{O},\sigma}]$, and J is generated by the r_i's. Then J will be called an $(\mathcal{O}, \sigma)$-*reduction ideal*, and $\mathcal{S}$ an $(\mathcal{O}, \sigma)$-*reduction set* of J.

Lemma 3.2

Let J be an $(\mathcal{O}, \sigma)$-*parametrizing ideal. Then the canonical homomorphism* $K[S_{c\mathcal{O},\sigma}]/J \longrightarrow K[\mathbf{x}, S_{c\mathcal{O},\sigma}]/(J + (g_1^*, \ldots, g_\eta^*))$ *makes the quotient ring* $K[\mathbf{x}, S_{c\mathcal{O},\sigma}]/(J + (g_1^*, \ldots, g_\eta^*))$ *into a free* $K[S_{c\mathcal{O},\sigma}]/J$-*module, and a basis is the set of the residue classes of the elements of* $\mathcal{O}$.

Proof. To prove this lemma we have to show that the residue clases of the elements in $\mathcal{O}$ generate $K[\mathbf{x}, S_{c\mathcal{O},\sigma}]/(J + (g_1^*, \ldots, g_\eta^*))$ and are linearly independent.

$\overline{\mathcal{O}}$ *generates.* It is enough to use the $\overline{\sigma}$-division algorithm, as we did in the proof of Theorem 2.8.

$\overline{\mathcal{O}}$ *is linearly independent over* $K[S_{c\mathcal{O},\sigma}]/J$. Suppose not. Then there would be a non-empty open set of $\mathrm{Spec}(K[S_{c\mathcal{O},\sigma}]/J)$, whose maximal linear ideals would represent ideals I of P for which $\mathcal{O}_\sigma(I) \subset \mathcal{O}$, a contradiction. $\square$

Remark 3.3   If J is an $(\mathcal{O}, \sigma)$-parametrizing ideal, then it is not necessarily an $(\mathcal{O}, \sigma)$-reduction ideal. It suffices to pick an ideal J which is an $(\mathcal{O}, \sigma)$-parametrizing ideal in $K[S_{c\mathcal{O},\sigma}]$ but not radical. Then $\sqrt{J}$ is still an $(\mathcal{O}, \sigma)$-parametrizing ideal but not necessarily an $(\mathcal{O}, \sigma)$-reduction ideal.

Lemma 3.4

Let $\mathbf{x} = x_1, \ldots, x_n$, $\mathbf{y} = y_1, \ldots, y_m$, *and let* $P = K[\mathbf{x}]$, $Q = K[\mathbf{x}, \mathbf{y}]$. *Let* $g_1, \ldots, g_t$ *be polynomials in* Q, *let* J *be the ideal generated by* $\{g_1, \ldots, g_t\}$, *and assume that there exist polynomials* $f_1(\mathbf{x}), \ldots, f_m(\mathbf{x})$ *such that the elements* $y_1 - f_1(\mathbf{x}), \ldots, y_m - f_m(\mathbf{x})$ *are in* J. *Then the ideal* $J \cap K[\mathbf{x}]$ *is generated by* $\{g_1(\mathbf{x}, \mathbf{f}), \ldots, g_t(\mathbf{x}, \mathbf{f})\}$ *where* $\mathbf{f} = (f_1, \ldots, f_m)$.

Proof. Every polynomial $g \in Q$ can be written as

$$g(\mathbf{x}, \mathbf{y}) = \sum_{i=1}^m h_i(y_i - f_i) + g(\mathbf{x}, \mathbf{f})$$

and the remainder $g(\mathbf{x}, \mathbf{f})$ is unique, since $\{y_1 - f_1(\mathbf{x}), \ldots, y_m - f_m(\mathbf{x})\}$ is a Gröbner basis with respect to an ordering which eliminates $\mathbf{y}$. Now the conclusion follows easily. $\square$

Proposition 3.5

The ideal $I(\mathbb{G}_{\mathcal{O},\sigma})$ is an $(\mathcal{O},\sigma)$-reduction ideal.

Proof. Following Definition 3.1 we have to prove that $I(\mathbb{G}_{\mathcal{O},\sigma})$ is an $(\mathcal{O},\sigma)$-parametrizing ideal, and that there exists an $(\mathcal{O},\sigma)$-reduction set of $I(\mathbb{G}_{\mathcal{O},\sigma})$. The first claim was proved in Corollary 2.11. To prove the second claim we use [13, Proposition 4.1] and [10, Section 4] to get generators of the ideal $I(\mathbb{B}_{\mathcal{O}})$ as the coefficients of the t_i's in polynomial expressions of type $\sum_{j=1}^{\nu} f_j g_j = \sum_{t_i \in \mathcal{O}} r_i t_i$ where the f_i's are polynomials in $K[\mathbf{x}, \mathbf{c}]$ and the r_i's are polynomials in $K[\mathbf{c}]$.

Consequently, to get generators of the ideal $L_{\mathcal{O},\sigma} + I(\mathbb{B}_{\mathcal{O}})$ we pick all these polynomial expressions, set equal to zero all the indeterminates which generate $L_{\mathcal{O},\sigma}$, and get expressions $\sum_{j=1}^{\nu} f_j^* g_j^* = \sum_{t_i \in \mathcal{O}} r_i^* t_i$ where the f_i^*'s are polynomials in $K[\mathbf{x}, S_{\mathcal{O},\sigma}]$ and the r_i^*'s are polynomials in $K[S_{\mathcal{O},\sigma}]$.

For the sake of clarity, let us call $\tilde{\mathbf{c}}$ the set of indeterminates $S_{c\mathcal{O},\sigma}$, and $\tilde{\mathbf{d}}$ the set of indeterminates $S_{\mathcal{O},\sigma} \setminus S_{c\mathcal{O},\sigma}$. We rewrite $\sum_{j=1}^{\nu} f_j^* g_j^* = \sum_{t_i \in \mathcal{O}} r_i^* t_i$

$$\sum_{j=1}^{\eta} f_j^*(\tilde{\mathbf{c}}, \tilde{\mathbf{d}}) \, g_j^*(\tilde{\mathbf{c}}) + \sum_{j=\eta+1}^{\nu} f_j^*(\tilde{\mathbf{c}}, \tilde{\mathbf{d}}) \, g_j^*(\tilde{\mathbf{d}}) = \sum_{t_i \in \mathcal{O}} r_i^*(\tilde{\mathbf{c}}, \tilde{\mathbf{d}}) \, t_i \,. \qquad (*)$$

Once more, we argue as in the proof of Theorem 2.9.d, and for every $j = \eta + 1, \ldots, \nu$ we produce elements $b_j - \sum_{\{i \mid b_j >_\sigma t_i\}} h_{ij} t_i$ which are in the ideal $(g_1^*, \ldots, g_\eta^*)$. Using Theorem 2.9, we get relations $c_{ij} = h_{ij}$ in the ring $K[\mathbf{c}]/(L_{\mathcal{O},\sigma} + I(\mathbb{B}_{\mathcal{O}}))$, in other words relations $c_{ij} - h_{ij} \in L_{\mathcal{O},\sigma} + I(\mathbb{B}_{\mathcal{O}})$. We make the substitution $c_{ij} \longrightarrow h_{ij}$ in the expressions $(*)$, write $\mathbf{h_{ij}}$ for the tuple h_{ij}, and get expressions

$$\sum_{j=1}^{\eta} \tilde{f}_j(\tilde{\mathbf{c}}) g_j^*(\tilde{\mathbf{c}}) = \sum_{t_i \in \mathcal{O}} \tilde{r}_i(\tilde{\mathbf{c}}) t_i \quad \text{where} \quad \tilde{r}_i(\tilde{\mathbf{c}}) = r_i^*(\tilde{\mathbf{c}}, \mathbf{h_{ij}}) \,. \qquad (**)$$

Now it suffices to prove that the set of all the $\tilde{r}_i(\tilde{\mathbf{c}})$ generates $I(\mathbb{G}_{\mathcal{O},\sigma})$. We recall the equality $I(\mathbb{G}_{\mathcal{O},\sigma}) = (L_{\mathcal{O},\sigma} + I(\mathbb{B}_{\mathcal{O}})) \cap K[S_{c\mathcal{O},\sigma}]$ and we know that the ideal $L_{\mathcal{O},\sigma} + I(\mathbb{B}_{\mathcal{O}})$ is generated by $L_{\mathcal{O},\sigma}$ and the polynomials $r_i^*(\tilde{\mathbf{c}}, \tilde{\mathbf{d}})$, hence the conclusion follows from the lemma. $\qquad \square$

Proposition 3.6

All the $(\mathcal{O}, \sigma)$-reduction ideals are equal.

Proof. Let J_1, J_2 be $(\mathcal{O}, \sigma)$-reduction ideals. By interchanging the role of J_1 and J_2 it suffices to prove that $J_1 \subseteq J_2$. Let $\mathcal{S}$ be an $(\mathcal{O}, \sigma)$-reduction set of J_1. Every element in $\mathcal{S}$ has the shape $\sum_{j=1}^{\eta} f_j g_j^* = \sum_{t_i \in \mathcal{O}} h_i t_i$. We consider the canonical homomorphism

$$K[S_{c\mathcal{O},\sigma}]/J_2 \longrightarrow K[\mathbf{x}, S_{c\mathcal{O},\sigma}]/(J_2 + (g_1^*, \ldots, g_\eta^*))$$

and deduce that $\sum_{t_i \in \mathcal{O}} h_i t_i = 0$ in the ring $K[\mathbf{x}, S_{c\mathcal{O},\sigma}]/(J_2 + (g_1^*, \ldots, g_\eta^*))$ which is free over $K[S_{c\mathcal{O},\sigma}]/J_2$ by Lemma 3.2. Therefore the coefficients h_i are zero in the ring $K[S_{c\mathcal{O},\sigma}]/J_2$. In particular, they belong to J_2 and the proof is complete. $\qquad \square$

A combination of Theorems 2.8 and 2.9 yields a remarkable property of $\mathbb{G}_{\mathcal{O},\sigma}$. A similar result can be found in [16, Proposition 4.3]. The main difference is that

there the authors deal with standard homogeneous saturated ideals. Moreover their proof is incorrect.

Corollary 3.7

Let $\mathcal{O} \subset \mathbb{T}^n$ be an order ideal of monomials, let σ be a term ordering on $\mathbb{T}^n$, and let $\mathbf{o}$ be the origin in the affine space $\mathbb{A}^{s(c\mathcal{O}, \sigma)}$.

a) *The point $\mathbf{o}$ belongs to $\mathbb{G}_{\mathcal{O}, \sigma}$.*
b) *The following conditions are equivalent*

    1) *The scheme $\mathbb{G}_{\mathcal{O}, \sigma}$ is isomorphic to an affine space.*
    2) *The point $\mathbf{o}$ is a smooth point of $\mathbb{G}_{\mathcal{O}, \sigma}$.*

Proof. The point $\mathbf{o}$ corresponds to the monomial ideal generated by $c\mathcal{O}$, and hence it belongs to $\mathbb{G}_{\mathcal{O}, \sigma}$ by Corollary 2.11. To prove part b), it is clearly sufficient to show that 2) implies 1). We argue as follows. Suppose that among the W-homogeneous generators of the ideal $I(\mathbb{G}_{\mathcal{O}, \sigma})$ there is one, say f, of type $c_{ij} - g$ with the property that c_{ij} does not divide any elements in the support of g. The graded ring $G_{\mathcal{O}, \sigma}/(f)$ is isomorphic to a graded K-algebra embedded in a polynomial ring with one less indeterminate, the isomorphism being constructed by substituting c_{ij} with g. Suppose we do this operation until no polynomial like f is found anymore, call Q/J the graded algebra obtained in this way, with Q a polynomial ring, and J a homogeneous ideal. We claim that no polynomial in J can have a non-zero linear part. For contradiction, suppose that a polynomial h of that type exists, and let c_{ij} be an indeterminate in the support of the linear part of h. Then c_{ij} must divide another power product in the support of h which is impossible since J is homogeneous with respect to a set of positive weights. In conclusion, we have $J = (0)$. $\qquad\square$

The algebraic argument given in the above proof agrees with the well-known fact that a quasi-cone over a projective subscheme $\mathbb{X}$ of a weighted projective scheme $\mathbb{P}(V)$ is smooth if and only if $\mathbb{X} = \mathbb{P}(V)$.

Remark 3.8 There is a strong difference between $\mathbb{G}_{\mathcal{O}, \sigma}$ and $\mathbb{B}_{\mathcal{O}, \sigma}$ even when $n = 2$. It is known that for $n = 2$ the scheme $\mathbb{B}_{\mathcal{O}}$ is smooth and irreducible. However, unlike the case of $\mathbb{G}_{\mathcal{O}, \sigma}$ as explained in Corollary 3.7, it does not need to be an affine cell (i.e. isomorphic to an affine space) as the following example shows.

EXAMPLE 3.9   This is an example where $\mathbb{G}_{\mathcal{O}, \sigma}$ is isomorphic to an affine space of dimension 9, and where $\mathbb{B}_{\mathcal{O}, \sigma}$ is a smooth irreducible variety of dimension 10 not isomorphic to an affine space. Let $P = k[x, y]$ and $\mathcal{O} = (1, x, y, x^2, y^2)$. Then $\partial\mathcal{O} = (xy, y^3, x^3, xy^2, x^2y)$ and so $\mu = \nu = 5$. Using CoCoA we compute $I(\mathbb{B}_{\mathcal{O}})$ and find out that $\dim(B_{\mathcal{O}}) = 10$. It is the expected number since the Hilbert scheme has only one component whose general point corresponds to the ideal of five distinct points in $\mathbb{A}^2$, and hence depends on ten parameters. Moreover we see that $\mathbb{B}_{\mathcal{O}}$ is isomorphic to a smooth irreducible variety of dimension 10, embedded in an affine space of dimension 14 and described by an ideal with 9 generators. Looking at the shape of the equations it is easy to see that it is not isomorphic to an affine space. The fact that $B_{\mathcal{O}}$ is smooth and irreducible agrees with a general statement that all the border basis

schemes in two indeterminates are smooth and irreducible (see [6, Proposition 2.4] and [9, Corollary 9.5.1]).

Now we let $\sigma = \mathtt{DegLex}$. We see that $S_{\mathcal{O},\sigma} = \{c_{11}, \ldots c_{55}\} \setminus \{c_{41}\}$ since $xy <_\sigma x^2$, hence $L_{\mathcal{O},\sigma} = (c_{41})$. Then we check that $c\mathcal{O} = (xy, y^3, x^3)$, hence $\eta = 3$, and L is the ideal generated by c_{41} in $K[c_{11}, \ldots, c_{53}]$. Now we check with CoCoA that the ring $B_{\mathcal{O}}/\overline{L}_{\mathcal{O},\sigma}$ is isomorphic to a polynomial ring with 9 indeterminates, and, in agreement with Corollary 2.11, we deduce from the Theorem 2.9 that also $G_{\mathcal{O},\sigma}$ is isomorphic to a polynomial ring with 9 indeterminates.

As a natural follow up to Theorem 2.9 we look for conditions under which we have $\overline{L}_{\mathcal{O},\sigma} = \emptyset$ so that diagram (2) can be written as

$$
\begin{array}{ccc}
G_{\mathcal{O},\sigma} & \overset{\varphi}{\simeq} & B_{\mathcal{O}} \\[2mm]
\Big\downarrow{\scriptstyle \Psi} & & \Big\downarrow{\scriptstyle \overline{\Phi}} \\[2mm]
U_{\mathcal{O},\sigma} & \overset{\vartheta}{\simeq} & U_{\mathcal{O}}
\end{array}
\tag{4}
$$

In other words we look for conditions under which the border basis scheme and the Gröbner basis scheme are isomorphic. We recall some definitions from [13, (Definition 2.7)] and [17].

DEFINITION 3.10 Let $\mathcal{O}$ be an order ideal, let V be a matrix in $\mathrm{Mat}_{1,n}(\mathbb{N}_+)$, and let σ be a term ordering on $\mathbb{T}^n$.

a) The order ideal $\mathcal{O}$ is said to have a *maxdeg$_V$ border* if $\deg_V(b) \geq \deg_V(t)$ for every $b \in c\mathcal{O}$ and every $t \in \mathcal{O}$.
b) Similarly, $\mathcal{O}$ is said to be a *V-cornercut* (or to have a *strong maxdeg$_V$ border*) if $\deg_V(b) > \deg_V(t)$ for every $b \in c\mathcal{O}$ and every $t \in \mathcal{O}$.
c) The order ideal $\mathcal{O}$ is said to be a *σ-cornercut* if $b >_\sigma t$ for every $b \in c\mathcal{O}$ and every $t \in \mathcal{O}$.

Proposition 3.11

Let $\mathcal{O}$ be an order ideal and σ a term ordering on $\mathbb{T}^n$. Consider the following conditions.

a_1) *The canonical embedding of $K[S_{\mathcal{O},\sigma}]$ in $K[c_{11}, \ldots, c_{\mu\nu}]$ induces an isomorphism between $G_{\mathcal{O},\sigma}$ and $B_{\mathcal{O}}$.*
a_2) *The canonical embedding of $K[\mathbf{x}, S_{\mathcal{O},\sigma}]$ in $K[\mathbf{x}, c_{11}, \ldots, c_{\mu\nu}]$ induces an isomorphism between $U_{\mathcal{O},\sigma}$ and $U_{\mathcal{O}}$.*
b_1) *The ideal $L_{\mathcal{O},\sigma}$ is the zero ideal.*
b_2) *The order ideal $\mathcal{O}$ is a σ-cornercut.*

Then a_1) is equivalent to a_2), b_1) is equivalent to b_2), and b_1) implies a_1).

Proof. The equivalence of a_1) and a_2) follows from Theorem 2.9, since $U_{\mathcal{O}}$ is a free $B_{\mathcal{O}}$ module with basis $\overline{\mathcal{O}}$, and also $U_{\mathcal{O},\sigma}$ is a free $G_{\mathcal{O},\sigma}$ module with basis $\overline{\mathcal{O}}$. Next we prove the implication b_1) $\implies$ b_2). If $L_{\mathcal{O},\sigma}$ is the zero ideal, then $b_j >_\sigma t_i$ for every

$j = 1, \ldots, \nu$ and every $i = 1, \ldots, \mu$. Consequently we have $b >_\sigma t$ for every $b \in c\mathcal{O}$ and every $t \in \mathcal{O}$ i.e. $\mathcal{O}$ is a σ-cornercut. The implication $b_2) \implies b_1)$ follows from the definition of $L_{\mathcal{O},\sigma}$ and the implication $b_1) \implies a_1)$ follows immediately from Theorem 2.9. □

Remark 3.12 Let us make some remarks about this proposition.

a) Remark 2.7 has the following implication. If condition $b_2)$ is fulfilled i.e. $\mathcal{O}$ is a σ-cornercut, then there exists a system V of positive weights such that $\mathcal{O}$ is a V-cornercut.

b) Example 3.9 shows that in the above proposition one cannot substitute condition $b_2)$ with the weaker condition that the order ideal $\mathcal{O}$ has a maxdeg$_V$-border.

c) The author does not know whether all the conditions of the above proposition are equivalent.

As a consequence of Proposition 3.11 we give a very short proof of the fact that if $\mathcal{O}$ has the shape of a segment then $B_{\mathcal{O}}$ is an affine space.

Corollary 3.13

Let $\mathcal{O} = \{1, x_n, x_n^2, \ldots, x_n^{\mu-1}\} \subset \mathbb{T}^n$. Then $\mathbb{B}_{\mathcal{O}}$ is isomorphic to the affine space $\mathbb{A}^{\mu n}$.

Proof. Clearly $\mathcal{O}$ is a Lex-cornercut, hence $B_{\mathcal{O}}$ is isomorphic to $G_{\mathcal{O},\text{Lex}}$ and we have $g_j^* = g_j$ for $j = 1, \ldots, \nu$. Corollary 2.11 implies that $G_{\mathcal{O},\text{Lex}}$ parametrizes all zero-dimensional ideals I in P for which $\mathcal{O} = \mathcal{O}_{\text{Lex}}(I)$. Hence $I(\mathbb{G}_{\mathcal{O},\text{Lex}})$ contains relations under which the generic Lex-Gröbner prebasis is the reduced Lex-Gröbner basis of an ideal I in P for which $\mathcal{O} = \mathcal{O}_{\text{Lex}}(I)$. On the other hand, it is clear that $\eta = n$ and the generic Lex-Gröbner prebasis consists of n polynomials whose leading terms are $x_1, \ldots, x_{n-1}, x_n^\mu$. They are pairwise coprime, hence every specialization of the generic Lex-Gröbner prebasis is a reduced Lex-Gröbner basis. It follows that $I(\mathbb{G}_{\mathcal{O}})$ is the zero ideal and the proof is complete. □

We observe that the explicit isomorphism of $B_{\mathcal{O}}$ with the polynomial ring $K[x_{11}, \ldots, x_{\mu n}]$ is given by expressing the indeterminates $x_{1,n+1}, \ldots, x_{\mu\nu}$ as polynomials in the indeterminates in $x_{11}, \ldots, x_{\mu n}$, as explained in the proof of Theorem 2.9.d.

The final part of the section and hence of the paper is devoted to a general remark and the discussion of some open problems.

Remark 3.14 In the paper [13] we have introduced and discussed the *homogeneous border basis scheme*. With the obvious modifications one can as well introduce the *homogeneous Gröbner basis scheme*.

Using Theorem 2.9 and Remark 2.10 we know the precise relation between the two schemes $\mathbb{G}_{\mathcal{O},\sigma}$ and $\mathbb{B}_{\mathcal{O}}$. It is then quite natural to ask the following question.

Question 1: *Is there any connection between the smoothness of the origin in $\mathbb{G}_{\mathcal{O},\sigma}$ and the smoothness of the origin in $\mathbb{B}_{\mathcal{O}}$?*

The scheme $\mathbb{G}_{\mathcal{O},\sigma}$ is connected since it is a quasi-cone, and hence all its points are connected to the origin (see Remark 2.12). Using Theorem 2.9 and Remark 2.10

we know the precise relation between the two schemes $\mathbb{G}_{\mathcal{O},\sigma}$ and $\mathbb{B}_{\mathcal{O}}$. However, the problem of the connectedness of $\mathbb{B}_{\mathcal{O}}$ is still open, so let me state it formally.

Question 2: *Is $\mathbb{B}_{\mathcal{O}}$ connected?*

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