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Collect. Math. **59**, 3 (2008), 325–346

Dynamical systems for rational normal curves

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Received September 13, 2007. Revised March 23, 2008

Abstract

We construct dynamical systems with $\mathbb{P}^n$ as state space, using $(n+1)\times(n+1)$ matrices of linear forms in $n+1$ variables, such that the fixed point sets are rational normal curves minus one point. Our matrices provide canonical forms for the triple action of PGL_{n+1} on the projective space of such matrices. Our dynamical systems include parameters identified with points in $\mathbb{P}^{n-1}$. We find conditions on these parameters to guarantee that any point in a dense open subset of $\mathbb{P}^n$ converges to a fixed point. We determine the domain of attraction of every fixed point.

1. Introduction

Let k be a field and V an n-dimensional vector space over k. The natural action of $GL(V)$ on $V \otimes V \otimes V$ poses a fundamental problem: Find invariants and canonical forms for its orbits, see [6]. The study of this problem might be called "trilinear algebra"; and the above action referred to as the "triple action" of $GL(V)$.

Choosing among the several possible presentations of $V \otimes V \otimes V$, we view its elements as $n \times n$ matrices of linear forms in n variables, see [2, 6]; and the triple action of $GL(V)$ as conjugation followed by a linear change of variables. Then it is convenient to pass to the projective space $\mathbb{P}(V \otimes V \otimes V)$ and to the projective group $G = PGL(V)$.

In this context, we produce matrices of linear forms, which provide canonical forms for their orbits under the triple action; and whose invariants are related to an associated quadratic rational map of projective space.

Keywords: rational map; canonical form; dynamical system.
MSC2000: 14E05; 14Q05; 15A72; 92D25.

After some preliminary material gathered in Section 2, in Section 3 we associate a dynamical system to a matrix of linear forms with parameters, then we prove the convergence to a fixed point for all points in a dense open subset of projective space, provided certain conditions are imposed on the parameters. We also determine the domain of attraction of every fixed point.

Our dynamical systems may qualify as discrete-time versions of the "toric systems" studied in [3]: In our systems, the fixed point sets are always standard rational normal curves minus one point; and the parameter space is a projective space. We answer a Global Attractor Conjecture for our systems.

In Section 4, we find that the stabilizer of any of our matrices is also the stabilizer of its associated rational map and that it consists of diagonal matrices. Thus, geometrically, the stabilizer may be viewed as a subset of the ambient projective space, where it is also a dense open subset of the standard rational normal curve. We also obtain that both our matrices and our maps are a cross-section for some orbits of the corresponding action of G.

Section 5 contains complementary information on quadratic rational maps similar to the ones previously discussed. We exhibit geometric and dynamic conditions, for lower dimensional cases, under which a quadratic rational map is in the orbit of our previously discussed maps. Section 6 informs the curious reader about the biological genesis of our dynamical systems.

The present work generalizes and improves our previous work [10]. The latter contains similar results up to dimension four, with less parameters; and with more computational ingredients. Both papers have origin in the author's work in population genetics, see [11, 12, 13]. All computations were performed using Macaulay2, see [4].

2. Preliminaries

Let k be the complex field, and let $R = k[x_0, x_1, ..., x_n]$ be the ring of polynomials in $n + 1$ variables over k. We write $M_{n+1}(R)$ for the ring of $(n + 1) \times (n + 1)$ matrices over R and $LF_{n+1}(R)$ for the subspace of matrices of linear forms.

Consider a nonzero matrix $A \in LF_{n+1}(R)$. Given a point in projective space $b \in \mathbb{P}^n$, we can "evaluate" A or any matrix representing its class $[A]$ in $\mathbb{P}(LF_{n+1})$, at b, written A_b, to obtain generically a rational map $\varphi_{A_b} : \mathbb{P}^n \to \mathbb{P}^n$, which sends any given point $c = (c_0, ..., c_n) \in \mathbb{P}^n$ to the point with homogeneous coordinates $(c_0, ..., c_n)A_b$. We sometimes write $A = A_x$, for the generic point $x = (x_0, ..., x_n)$; using this notation, the matrix A has a canonically associated quadratic rational map $\varphi_A : \mathbb{P}^n \to \mathbb{P}^n$ given by $\varphi_A(x) = (x_0, ..., x_n)A_x$.

If an algebraic group G acts on a variety X and we are given elements $g \in G$ and $x \in X$, we write $g \cdot x$ or else x^g for the translate of x with g, G^x for the stabilizer of x in G and $G \cdot x$ for the orbit of x, see [1, 7, 9].

Let n be a fixed positive integer, and let G denote the projective general linear group $PGL_{n+1}(k)$, identified with the $(n+1)\times(n+1)$ invertible matrices over k modulo the scalars. We now define the triple action of G on the space $\mathbb{P}(LF_{n+1})$. Given $g \in G$ and $A \in LF_{n+1}(R)$ representing an element $[A] \in \mathbb{P}(LF_{n+1})$, we choose an invertible matrix $B \in M_{n+1}(k)$ representing g. This produces the translate $[A]^g$: represented by the matrix obtained from BAB^{-1} replacing in its entries each x_i for $i = 0, ..., n$ by the i-th coordinate of the row vector xB.

There is another action of the group G on the space Φ of quadratic rational maps $\mathbb{P}^n \to \mathbb{P}^n$ given by conjugation. We say that $\varphi, \psi \in \Phi$ are *projectively equivalent* when they are elements of the same orbit under this action.

The association $\mathbb{P}(LF_{n+1}) \to \Phi$ is not injective, but we have a surjective linear map $\Psi : LF_{n+1}(R) \to Q_{n+1}$ from the space of matrices of linear forms to the space Q_{n+1} of $1 \times (n+1)$ matrices of quadratic forms in R, given by left multiplication with the row vector x, whose kernel must be considered. We define the *polynomial elementary column* E_{ij} of size $n+1$, for any given pair of indices $0 \leq i < j \leq n$, to be $(0, ..., 0, -x_j, 0, ..., 0, x_i, 0, ..., 0)^t$, where $-x_j$ appears in the i-th position and x_i appears in the j-th position.

We will use the following results from [10]:

Theorem 2.1

Let k be a field, $R = k[x_0, x_1, ..., x_n], 0 \neq A \in LF_{n+1}(R)$ and let $\varphi : \mathbb{P}^n \to \mathbb{P}^n$ be given by $\varphi(x) = xA$. If $\mathcal{L} : \mathbb{P}^n \to \mathbb{P}^n$ is the projective map given by $\mathcal{L}(x) = xB$ with $[B] \in G = PGL_{n+1}(k)$, then the composition $\mathcal{L}^{-1} \circ \varphi \circ \mathcal{L} : \mathbb{P}^n \to \mathbb{P}^n$ is given by $(\mathcal{L}^{-1} \circ \varphi \circ \mathcal{L})(x) = xA^{[B]}$.

Proposition 2.2

$\ker \Psi$ *is the set of matrices in $LF_{n+1}(R)$ whose columns are in the span of $\{E_{ij} \,|\, i < j\}$.*

3. The dynamical systems

Let $a = (a_0, ..., a_n) \in \mathbb{P}^n$ be a vector of parameters. The vector $a \in \mathbb{P}^n$ will be called generic in this paper whenever

$$\prod_{i \neq n-1} a_i \prod_{i<j}^{i,j \neq n-1} (a_i - a_j) \neq 0. \tag{3.1}$$

We will have occasion to use the catalecticant matrix

$$C = \begin{pmatrix} x_0 & x_1 & \cdots & x_{n-2} & x_{n-1} \\ x_1 & x_2 & \cdots & x_{n-1} & x_n \end{pmatrix}; \tag{3.2}$$

and its following minors

$$c_i = \begin{vmatrix} x_i & x_{n-1} \\ x_{i+1} & x_n \end{vmatrix}, \ 0 \leq i \leq n-2. \tag{3.3}$$

Consider the matrix of linear forms

$$T_a = \begin{pmatrix} a_0 x_n & 0 & \cdots & 0 & 0 \\ (a_n - a_0) x_{n-1} & a_1 x_n & \cdots & 0 & 0 \\ & \ddots & \ddots & & \\ 0 & 0 & \cdots & a_{n-1} x_n & 0 \\ 0 & 0 & \cdots & (a_n - a_{n-1}) x_{n-1} & a_n x_n \end{pmatrix}. \tag{3.4}$$

For a generic point a, we have a rational map $\varphi_a : \mathbb{P}^n \to \mathbb{P}^n$ given by $\varphi_a(x) = xT_a$, for $x = (x_0, ..., x_n) \in \mathbb{P}^n$. The iteration of φ_a will originate our dynamical system. As a first step, we proceed to determine the fundamental and fixed points of φ_a.

Theorem 3.1

For a generic point a, φ_a has the following properties:

(1) *The determinantal algebraic set of fixed and fundamental points of the map, defined by the 2×2 minors from the $2 \times (n+1)$ matrix with rows x and $\varphi_a(x)$, has support $A \cup L$, where A is the standard rational normal curve in $\mathbb{P}^n$; and $L = Z(x_{n-1}, x_n)$ is the tangential linear space to A of codimension two at the point $Q = (1, 0, ..., 0) \in A$.*
(2) *The fundamental points of the map have support equal to L.*
(3) *The set of fixed points of the map is $A \setminus Q$.*

Proof. We write F for the set of fundamental points of φ_a; and $(y_0, ..., y_n) = \varphi_a(x_0, ..., x_n)$. The definition of T_a guarantees that $L = Z(x_{n-1}, x_n) \subseteq F$.

Observe that $y_n = a_n x_n^2$ while $y_{n-2} = a_{n-2}x_{n-2}x_n + (a_n - a_{n-2})x_{n-1}^2$. Thus, $a_n, a_n - a_{n-2} \neq 0$ implies $F \subseteq Z(x_{n-1}, x_n)$. This proves (2).

The algebraic set of fixed and fundamental points of φ_a is determinantal, given by the ideal $\mathfrak{a}$ generated by the 2×2 minors from the following $2 \times (n+1)$ matrix, whose rows are x and $\varphi_a(x)$:

$$\begin{pmatrix} x_0 & x_1 & \cdots & x_n \\ a_0x_0x_n + (a_n - a_0)x_1x_{n-1} & a_1x_1x_n + (a_n - a_1)x_2x_{n-1} & \cdots & a_nx_n^2 \end{pmatrix}.$$

Among these minors, we have

$$\begin{vmatrix} x_{n-1} & x_n \\ a_{n-1}x_{n-1}x_n + (a_n - a_{n-1})x_{n-1}x_n & a_nx_n^2 \end{vmatrix} = \begin{vmatrix} x_{n-1} & x_n \\ a_nx_{n-1}x_n & a_nx_n^2 \end{vmatrix} = 0,$$

as well as

$$\begin{vmatrix} x_i & x_n \\ a_ix_ix_n + (a_n - a_i)x_{i+1}x_{n-1} & a_nx_n^2 \end{vmatrix} = (a_n - a_i)x_n \begin{vmatrix} x_i & x_{n-1} \\ x_{i+1} & x_n \end{vmatrix};$$

and

$$\begin{vmatrix} x_i & x_{n-1} \\ a_ix_ix_n + (a_n - a_i)x_{i+1}x_{n-1} & a_nx_{n-1}x_n \end{vmatrix} = (a_n - a_i)x_{n-1} \begin{vmatrix} x_i & x_{n-1} \\ x_{i+1} & x_n \end{vmatrix},$$

for $0 \leq i \leq n-2$. Since $a_n - a_i \neq 0$ for these indices, the ideal

$$\mathfrak{b} = \text{saturate}\,(\mathfrak{a}, (x_{n-1}, x_n)) = \bigcup_{\nu=1}^{\infty} (\mathfrak{a} : (x_{n-1}, x_n)^\nu)\ ,$$

which describes the closure of the set of fixed points of φ_a, contains all minors c_i, defined in (3.3), of the catalecticant C, for $0 \leq i \leq n-2$.

The remaining generators of $\mathfrak{a}$ are

$$\begin{vmatrix} x_i & x_j \\ a_i x_i x_n + (a_n - a_i) x_{i+1} x_{n-1} & a_j x_j x_n + (a_n - a_j) x_{j+1} x_{n-1} \end{vmatrix}$$

$$= (a_n - a_i) x_j \begin{vmatrix} x_i & x_{n-1} \\ x_{i+1} & x_n \end{vmatrix} - (a_n - a_j) x_i \begin{vmatrix} x_j & x_{n-1} \\ x_{j+1} & x_n \end{vmatrix},$$

for all $0 \le i < j \le n-2$.

Thus, for generic parameters, the saturation with respect to (x_{n-1}, x_n), of the ideal generated by the minors c_i, already generates $\mathfrak{b}$.

Let $\mathfrak{c}$ be the ideal generated by all 2×2 minors from the catalecticant matrix (3.2). It is well known, see [2, 6, 8], that $\mathfrak{c}$ is the prime ideal of the standard rational normal curve A.

Now, $\mathfrak{c}$ contains the minors c_i. Hence $\mathfrak{c}$ also contains the saturation with respect to the ideal (x_{n-1}, x_n) of the ideal that these c_i's generate. It follows that $\mathfrak{b} \subseteq \mathfrak{c}$.

On the other hand, $\mathfrak{c} \subseteq \mathfrak{b}$, because the identities

$$x_n \begin{vmatrix} x_i & x_j \\ x_{i+1} & x_{j+1} \end{vmatrix} = x_{j+1} \begin{vmatrix} x_i & x_{n-1} \\ x_{i+1} & x_n \end{vmatrix} - x_{i+1} \begin{vmatrix} x_j & x_{n-1} \\ x_{j+1} & x_n \end{vmatrix},$$

$$x_{n-1} \begin{vmatrix} x_i & x_j \\ x_{i+1} & x_{j+1} \end{vmatrix} = x_j \begin{vmatrix} x_i & x_{n-1} \\ x_{i+1} & x_n \end{vmatrix} - x_i \begin{vmatrix} x_j & x_{n-1} \\ x_{j+1} & x_n \end{vmatrix},$$

which are valid for all $0 \le i < j \le n-2$, express most minors in $\mathfrak{c}$ in (saturation acceptable) terms of other minors in $\mathfrak{c}$, already identified as elements of $\mathfrak{b}$ by earlier identities.

Thus, $\mathfrak{b}$ is the ideal generated by all 2×2 minors from C. This proves (1) and (3). $\square$

Each minor c_j of the catalecticant (3.2) is a quadratic form. We wish to compare these forms with the quartic forms resulting from substitution of y_i instead of x_i for all $i = 0, ..., n$, where $y = \varphi_a(x)$.

Lemma 3.2

The minors c_j satisfy the polynomial identity

$$c_j(\varphi_a(x)) = a_j a_n x_n^2 c_j - (a_{j+1} - a_n) a_n x_{n-1} x_n c_{j+1}.$$

Proof.

$$\begin{aligned} c_j(\varphi_a(x)) &= \begin{vmatrix} a_j x_j x_n + (a_n - a_j) x_{j+1} x_{n-1} & a_n x_{n-1} x_n \\ a_{j+1} x_{j+1} x_n + (a_n - a_{j+1}) x_{j+2} x_{n-1} & a_n x_n^2 \end{vmatrix} \\ &= \begin{vmatrix} a_j c_j + a_n x_{j+1} x_{n-1} & a_n x_{n-1} x_n \\ a_{j+1} c_{j+1} + a_n x_{j+2} x_{n-1} & a_n x_n^2 \end{vmatrix} \\ &= a_n x_n \begin{vmatrix} a_j c_j & x_{n-1} \\ a_{j+1} c_{j+1} & x_n \end{vmatrix} + a_n^2 x_{n-1} x_n \begin{vmatrix} x_{j+1} & x_{n-1} \\ x_{j+2} & x_n \end{vmatrix} \end{aligned}$$

$$
\begin{aligned}
&= a_j a_n x_n^2 c_j - a_{j+1} a_n x_{n-1} x_n c_{j+1} + a_n^2 x_{n-1} x_n c_{j+1} \\
&= a_j a_n x_n^2 c_j - (a_{j+1} - a_n) a_n x_{n-1} x_n c_{j+1}. \qquad (3.5)
\end{aligned}
$$

□

Dynamical systems

The systems under study are discrete-time, they are generated by the iteration of rational maps $\varphi_a : \mathbb{P}^n \to \mathbb{P}^n$ for generic parameter vectors $a \in \mathbb{P}^n$. Thus, the state space is projective space. The next theorem shows that under adequate conditions on the parameters, any generic point of the state space converges to a fixed point.

Theorem 3.3

Let $H = Z(x_n)$ and $L = Z(x_{n-1}, x_n)$. For a generic point a, the dynamical system obtained by the iteration of φ_a has the following properties:

(1) $\varphi_a(H \smallsetminus L) \subseteq L$.

(2) *The polynomial eigenvalue $E_n = \ker(T_a - a_n x_n)$ parametrizes the standard rational normal curve A as follows:*

$$E_n = (x_{n-1}^n, x_{n-1}^{n-1} x_n, ..., x_{n-1} x_n^{n-1}, x_n^n).$$

(3) *The polynomial eigenvalue $E_i = \ker(T_a - a_i x_n)$ parametrizes the rational normal curve (of degree i), $E_i = (e_{i0}, ..., e_{ii}, 0, ..., 0)$ in a space isomorphic to $\mathbb{P}^i$, which is the linear component of dimension i in the osculating flag to A at Q, for $i = 0, ..., n-1$. Here,*

$$e_{i\nu} = \prod_{j=0}^{\nu-1} (a_j - a_i) \prod_{j=\nu}^{i-1} (a_j - a_n) x_{n-1}^{i-\nu} x_n^{\nu}, \ 0 \le \nu \le i. \qquad (3.6)$$

(4) *Using the parametrizations above, we have the polynomial identity*

$$p_n x_n^{n-1} x = \sum_{i=0}^{n} p_i E_i, \qquad (3.7)$$

where $p_n = \prod_{\mu<\nu<n-1} (a_\mu - a_\nu)$, $p_{n-1} = 0$; and p_i is the following homogeneous polynomial of degree $n - i$, for $i = 0, ..., n-2$:

$$
\begin{aligned}
p_i = &\prod_{\mu<\nu<n-1}^{\mu,\nu\neq i} (a_\mu - a_\nu) \\
&\times \sum_{j=i}^{n-2} (-1)^{i+j} \prod_{\lambda=j+1}^{n-2} (a_i - a_\lambda) \prod_{\lambda=i+1}^{j} (a_\lambda - a_n) x_{n-1}^{j-i} x_n^{n-2-j} c_j. \qquad (3.8)
\end{aligned}
$$

(5) *We also have the polynomial identity $p_i[\varphi_a(x)] = a_i a_n^{n-i-1} x_n^{n-i} p_i(x)$, for each $i = 0, ..., n-2$.*

(6) *Every point outside the hyperplane H converges to a fixed point, provided $|a_i| < |a_n|$ for all $i = 0, ..., n-2$.*

Proof. If $x_n = 0$, but $x_{n-1} \neq 0$, then the following equality proves (1):

$$\varphi_a(x) = ((a_n - a_0)x_1, ..., (a_n - a_{n-2})x_{n-1}, 0, 0).$$

Next, we wish to calculate the left-hand kernel of the matrix $T_a - a_n x_n$. This will be a point in $\mathbb{P}^n_R$, which we will interpret as a single parametrization of a set of points in $\mathbb{P}^n_k$. It is easy to verify that the polynomial eigenvalue E_n is the left-hand kernel of the matrix

$$\begin{pmatrix} -x_n & 0 & \cdots & 0 & 0 \\ x_{n-1} & -x_n & \cdots & 0 & 0 \\ & \ddots & \ddots & & \\ 0 & 0 & \cdots & -x_n & 0 \\ 0 & 0 & \cdots & x_{n-1} & 0 \end{pmatrix}.$$

The solutions in the ring of polynomials R of this system of linear equations are the following, proving (2):

$$E_n = (x_{n-1}^n, x_{n-1}^{n-1}x_n, ..., x_{n-1}x_n^{n-1}, x_n^n). \tag{3.9}$$

In a similar way, if $0 \leq i \leq n-1$, our search for the polynomial eigenvalue $E_i = \ker(T_a - a_i x_n)$ leads us to the left-hand kernel of the matrix

$$\begin{pmatrix} (a_0 - a_i)x_n & 0 & \cdots & 0 & 0 \\ (a_n - a_0)x_{n-1} & (a_1 - a_i)x_n & \cdots & 0 & 0 \\ & \ddots & \ddots & & \\ 0 & 0 & \cdots & (a_{n-1} - a_i)x_n & 0 \\ 0 & 0 & \cdots & (a_n - a_{n-1})x_{n-1} & (a_n - a_i)x_n \end{pmatrix}.$$

The solutions in the ring of polynomials R of this system of linear equations are the stated ones, where each linear space $L_i = Z(x_{i+1}, x_{i+2}, ..., x_n) \cong \mathbb{P}^i_k$, for $0 \leq i \leq n-1$, is tangent to A at Q, because of the set-theoretic equality $A \cap L_i = \{Q\}$.

Observe that for $0 \leq i \leq n-1$, the eigenvalue E_i is a single point $y = (y_0, ..., y_i, 0, ..., 0) \in \mathbb{P}^n_R$, interpreted as a single parametrization for the points in $\mathbb{P}^i_k$, satisfying the ideal generated by all 2×2 minors of the matrix

$$C_i = \begin{pmatrix} (a_0 - a_i)y_0 & \cdots & (a_{i-1} - a_i)y_{i-1} & x_{n-1} \\ (a_0 - a_n)y_1 & \cdots & (a_{i-1} - a_n)y_i & x_n \end{pmatrix}.$$

This is the stated rational normal curve of $\mathbb{P}^i_k$, proving (3).

We start the verification of (4), observing that the last two coordinates on one side of the asserted equality are indeed equal to the corresponding ones on the other side.

In lieu of a complete proof, we offer a verification of the equality of the $(n-2)$nd and $(n-3)$rd coordinates for (3.7), assuming that n is big enough.

The left-hand side of Equation (3.7) has the following $(n-2)$nd coordinate:

$$\prod_{\mu<\nu<n-1} (a_\mu - a_\nu)x_{n-2}x_n^{n-1},$$

whereas the corresponding right-hand side coordinate is:

$$\begin{aligned}
&\prod_{\mu<\nu<n-2}(a_\mu-a_\nu)c_{n-2}e_{n-2,n-2} + \prod_{\mu<\nu<n-1}(a_\mu-a_\nu)e_{n,n-2}\\
&=\prod_{\mu<\nu<n-2}(a_\mu-a_\nu)(x_{n-2}x_n-x_{n-1}^2)\prod_{j=0}^{n-3}(a_j-a_{n-2})x_n^{n-2}\\
&\quad+\prod_{\mu<\nu<n-1}(a_\mu-a_\nu)x_{n-1}^2x_n^{n-2}=\prod_{\mu<\nu<n-1}(a_\mu-a_\nu)x_{n-2}x_n^{n-1}.
\end{aligned}$$

The left-hand side of Equation (3.7) has the following $(n-3)$rd coordinate:

$$\prod_{\mu<\nu<n-1}(a_\mu-a_\nu)x_{n-3}x_n^{n-1},$$

whereas the corresponding right-hand side coordinate is:

$$\begin{aligned}
&\prod_{\mu<\nu<n-1}^{\mu,\nu\neq n-3}(a_\mu-a_\nu)(a_{n-3}-a_{n-2})x_nc_{n-3}e_{n-3,n-3}\\
&\quad-\prod_{\mu<\nu<n-1}^{\mu,\nu\neq n-3}(a_\mu-a_\nu)(a_{n-2}-a_n)x_{n-1}c_{n-2}e_{n-3,n-3}\\
&\quad+\prod_{\mu<\nu<n-2}(a_\mu-a_\nu)c_{n-2}e_{n-2,n-3}+\prod_{\mu<\nu<n-1}(a_\mu-a_\nu)e_{n,n-3}\\
&=\prod_{\mu<\nu<n-1}(a_\mu-a_\nu)x_nc_{n-3}x_n^{n-3}\\
&\quad-\prod_{\mu<\nu<n-1}^{\mu,\nu\neq n-3}(a_\mu-a_\nu)(a_{n-2}-a_n)x_{n-1}c_{n-2}x_n^{n-3}\\
&\quad+\prod_{\mu<\nu<n-2}(a_\mu-a_\nu)c_{n-2}\prod_{\lambda=0}^{n-4}(a_\lambda-a_{n-2})(a_{n-3}-a_n)x_{n-1}x_n^{n-3}\\
&\quad+\prod_{\mu<\nu<n-1}(a_\mu-a_\nu)x_{n-1}^3x_n^{n-3}\\
&=\prod_{\mu<\nu<n-1}(a_\mu-a_\nu)c_{n-3}x_n^{n-2}\\
&\quad+\prod_{\mu<\nu<n-2}(a_\mu-a_\nu)c_{n-2}\prod_{\lambda=0}^{n-4}(a_\lambda-a_{n-2})(a_{n-3}-a_{n-2})x_{n-1}x_n^{n-3}\\
&\quad+\prod_{\mu<\nu<n-1}(a_\mu-a_\nu)x_{n-1}^3x_n^{n-3}\\
&=\prod_{\mu<\nu<n-1}(a_\mu-a_\nu)(x_{n-3}x_n-x_{n-2}x_{n-1})x_n^{n-2}\\
&\quad+\prod_{\mu<\nu<n-1}(a_\mu-a_\nu)(x_{n-2}x_n-x_{n-1}^2)x_{n-1}x_n^{n-3}+\prod_{\mu<\nu<n-1}(a_\mu-a_\nu)x_{n-1}^3x_n^{n-3}\\
&=\prod_{\mu<\nu<n-1}(a_\mu-a_\nu)x_{n-3}x_n^{n-1}.
\end{aligned}$$

In order to prove (5), let us write

$$B_{i,j} = (-1)^{i+j} \prod_{\lambda=j+1}^{n-2} (a_i - a_\lambda) \prod_{\lambda=i+1}^{j} (a_\lambda - a_n) x_{n-1}^{j-i} x_n^{n-2-j}.$$

Equation (3.8) becomes:

$$p_i = \prod_{\mu<\nu<n-1}^{\mu,\nu\neq i} (a_\mu - a_\nu) \sum_{j=i}^{n-2} B_{i,j} c_j.$$

Lemma (3.2) implies that

$$(B_{i,j}c_j)(\varphi_a(x)) = a_j a_n^{n-i-1} x_n^{n-i} B_{i,j} c_j + (a_i - a_{j+1}) a_n^{n-i-1} x_n^{n-i} B_{i,j+1} c_{j+1};$$

and this equality immediately implies (5).

The last assertion (6) follows immediately from (4) and (5): If we compare the expression (3.7) for x with the corresponding one for $\varphi_a(x)$, we obtain the asserted convergence: $\lim_{m\to\infty} \varphi_a^m(x) = E_n(x) \in A$, for x in the affine space $x_n = 1$, because the coefficient p_n dominates the other coefficients under the hypothesis of (6). □

Remark 3.4 The parameter a_{n-1} plays no role towards the definition of the map φ_a from the matrix T_a, and it can be omitted. In fact, we could write $a_{n-1} = 0$ in the definition of T_a, preserving the properties of φ_a and T_a.

The next corollaries follow immediately from the proof above. They complete the description of the dynamical systems: They tell us where a generic point goes and what is the domain of attraction of a given fixed point.

Corollary 3.5

If $x \notin H$, then $\lim_{m\to\infty} \varphi_a^m(x) = E_n(x)$.

Corollary 3.6

Given $y = (y_0, ..., y_n) \in A \smallsetminus \{Q\}$, the set of all $x \in \mathbb{P}^n$ satisfying $\lim_{m\to\infty} \varphi_a^m(x) = y$ is locally closed, described by the conditions $x_n \neq 0$ and

$$\begin{vmatrix} x_{n-1} & y_{n-1} \\ x_n & y_n \end{vmatrix} = 0.$$

Remark 3.7 *The global attractor conjecture for our dynamical systems.* We have a stratification for an open subset of our state space $\mathbb{P}^n$ according to points $\beta = (t, u) \in \mathbb{P}^1$: To each point β we associate the hyperplane H_β defined by the equation $tx_n - ux_{n-1} = 0$. The hyperplane $H_{(1,0)}$ is special. Let us call the remaining strata "generic".

We have identified each generic hyperplane H_β as the domain of attraction of a fixed point. Furthermore, any generic H_β is stable under any of our generic rational maps φ_α, as seen at the beginning of the proof of Theorem 3.1. Also, H_β intersects the fixed point set at exactly one point.

We consider that all of the above is the geometric part of the hypothesis of the conjecture. It is satisfied by the dynamical system associated to any of our rational maps φ_α. Having an adequate geometry present, we exhibited the Lyapunov-type conditions in Theorem 3.3 (6) that guarantee the desired convergence; and the truth of the Conjecture. Thus, the geometry does not entirely determine the dynamics. It is only an important ingredient.

EXAMPLE 3.8 *The rational normal Quintic.* Consider the matrix

$$T_a = \begin{pmatrix} a_0x_5 & 0 & & 0 & 0 \\ (a_5-a_0)x_4 & a_1x_5 & & 0 & 0 \\ 0 & (a_5-a_1)x_4 & \ddots & 0 & 0 \\ 0 & 0 & \ddots & a_4x_5 & 0 \\ 0 & 0 & & (a_5-a_4)x_4 & a_5x_5 \end{pmatrix}.$$

Its eigenvalues $E_i = \ker(T_a - a_i x_5)$ may be parametrized as follows:

$$\begin{aligned} E_5 &= (x_4^5, x_4^4x_5, x_4^3x_5^2, x_4^2x_5^3, x_4x_5^4, x_5^5); \\ E_4 &= ((a_0-a_5)(a_1-a_5)(a_2-a_5)(a_3-a_5)x_4^4, \\ &\quad (a_0-a_4)(a_1-a_5)(a_2-a_5)(a_3-a_5)x_4^3x_5, \\ &\quad (a_0-a_4)(a_1-a_4)(a_2-a_5)(a_3-a_5)x_4^2x_5^2, \\ &\quad (a_0-a_4)(a_1-a_4)(a_2-a_4)(a_3-a_5)x_4x_5^3, \\ &\quad (a_0-a_4)(a_1-a_4)(a_2-a_4)(a_3-a_4)x_5^4, 0); \\ E_3 &= ((a_0-a_5)(a_1-a_5)(a_2-a_5)x_4^3, (a_0-a_3)(a_1-a_5)(a_2-a_5)x_4^2x_5, \\ &\quad (a_0-a_3)(a_1-a_3)(a_2-a_5)x_4x_5^2, (a_0-a_3)(a_1-a_3)(a_2-a_3)x_5^3, 0, 0); \\ E_2 &= ((a_0-a_5)(a_1-a_5)x_4^2, (a_0-a_2)(a_1-a_5)x_4x_5, \\ &\quad (a_0-a_2)(a_1-a_2)x_5^2, 0, 0, 0); \\ E_1 &= ((a_0-a_5)x_4, (a_0-a_1)x_5, 0, 0, 0, 0); \\ E_0 &= (1, 0, 0, 0, 0, 0). \end{aligned}$$

We have the identity

$$p_5x_5^4x = p_0E_0 + p_1E_1 + p_2E_2 + p_3E_3 + p_5E_5, \tag{3.10}$$

where

$$\begin{aligned} p_0 &= \prod_{1\le i<j\le 3}(a_i-a_j) \\ &\quad \times [(a_0-a_1)(a_0-a_2)(a_0-a_3)x_5^3c_0 - (a_0-a_2)(a_0-a_3)(a_1-a_5)x_4x_5^2c_1 \\ &\qquad + (a_0-a_3)(a_1-a_5)(a_2-a_5)x_4^2x_5c_2 - (a_1-a_5)(a_2-a_5)(a_3-a_5)x_4^3c_3], \end{aligned}$$

$$p_1 = \prod_{i<j}^{i,j\in\{0,2,3\}} (a_i - a_j)$$
$$\times [(a_1 - a_2)(a_1 - a_3)x_5^2 c_1 - (a_1 - a_3)(a_2 - a_5)x_4 x_5 c_2$$
$$+ (a_2 - a_5)(a_3 - a_5)x_4^2 c_3],$$

$$p_2 = \prod_{i<j}^{i,j\in\{0,1,3\}} (a_i - a_j)[(a_2 - a_3)x_5 c_2 - (a_3 - a_5)x_4 c_3],$$
$$p_3 = \prod_{0\le i<j\le 2} (a_i - a_j)c_3, \quad p_5 = \prod_{0\le i<j\le 3} (a_i - a_j);$$

where we have written c_i for the following minors of the matrix C:

$$C = \begin{pmatrix} x_0 & x_1 & x_2 & x_3 & x_4 \\ x_1 & x_2 & x_3 & x_4 & x_5 \end{pmatrix}; \; c_i = \begin{vmatrix} x_i & x_4 \\ x_{i+1} & x_5 \end{vmatrix}, \text{ for } 0 \le i \le 3.$$

Finally, if $x = (x_0, ..., x_5)$ and $\varphi(x) = xT_a$, then

$$p_0[\varphi(x)] = a_0 a_5^4 x_5^5 p_0, \; p_1[\varphi(x)] = a_1 a_5^3 x_5^4 p_1,$$
$$p_2[\varphi(x)] = a_2 a_5^2 x_5^3 p_2, \; p_3[\varphi(x)] = a_3 a_5 x_5^2 p_3.$$

Remark 3.9 For any generic parameter point a, the transformation φ_a is a Cremona map, with inverse rational map ψ_a defined on $\mathbb{P}^n \smallsetminus Z(x_{n-1}, x_n)$ by $\psi_a(x) = xJ_a$, where $[J_a]$ admits as a representative the transpose of the cofactor matrix of T_a.

We will now use the following catalecticant matrix and its 2×2 minors:

$$C(i, n) = \begin{pmatrix} x_i & x_{i+1} & \cdots & x_{n-1} \\ x_{i+1} & x_{i+2} & \cdots & x_n \end{pmatrix}.$$

Corollary 3.10

Let A be the standard rational normal curve of degree n in $\mathbb{P}^n$. Let $Q = W_0 \subset W_1 \subset \cdots \subset W_n = \mathbb{P}^n$ be the osculating flag to A at $Q = (1, 0, ..., 0)$. Then the join $J(A, W_i)$ has homogeneous ideal generated by the 2×2 minors of the catalecticant matrix $C(i + 1, n)$.

Proof. Let $\mathfrak{a}_i$ be the ideal generated by the 2×2 minors of $C(i, n)$, for $0 \le i \le n$. Since these ideals are well known to be prime, see [2, 6, 8], it will be enough to prove that $Z(\mathfrak{a}_{i+1}) = J(A, W_i)$.

Equations (3.8) imply that $p_j \in \mathfrak{a}_i$, for $j \ge i, j \ne n$.

Equations (3.7) imply that $Z(\mathfrak{a}_{i+1}) \subseteq J(A, W_i)$, because $E_i \subseteq W_i$.

Conversely, we have that each $J(A, W_i)$ is irreducible; and furthermore, $J(A, W_i) \nsubseteq W_{n-2} = Z(x_{n-1}, x_n)$. If $b \in J(A, W_i) \smallsetminus Z(x_{n-1}, x_n)$, then the set of vectors $E = \{E_0(b), ..., E_{n-2}(b)\}$ is linearly independent; and (3.7) gives rise to the unique expression of the vector $p_n(b)b_n^{n-1}b - p_n(b)E_n(b)$ as a linear combination of E.

Thus, $p_j(b) = 0$, for $j > i$; and b satisfies the saturation with respect to the ideal (x_{n-1}, x_n) of the ideal generated by the c_j with $j > i$. This means that $b \in Z(\mathfrak{a}_{i+1})$. Hence $J(A, W_i) \subseteq Z(\mathfrak{a}_{i+1})$. □

Corollary 3.11

For the standard rational normal curve A in $\mathbb{P}^n$ and its osculating flag $W_0 \subset W_1 \subset \cdots \subset W_n$ at some point $W_0 \in A$, we have

(1) $\dim J(A, W_i) = i + 2$.
(2) *In particular,* $J(A, W_{n-2}) = \mathbb{P}^n$.
(3) $\deg J(A, W_i) = n - i - 1$.

EXAMPLE 3.12 For the rational normal quintic, let $M_2(i, 5)$ be the ideal generated by all 2×2 minors of

$$C(i,5) = \begin{pmatrix} x_i & x_{i+1} & \cdots & x_4 \\ x_{i+1} & x_{i+2} & \cdots & x_5 \end{pmatrix}.$$

In the polynomial ring $k[a_0, ..., a_5, x_0, ..., x_5]$, we have the primary decompositions

$$\begin{aligned}
\sqrt{\text{saturate } ((p_3), p_5)} &= (x_4, x_5) \cap M_2(3,5), \\
\sqrt{\text{saturate } ((p_2, p_3), p_5)} &= (x_4, x_5) \cap M_2(2,5), \\
\sqrt{\text{saturate } ((p_1, p_2, p_3), p_5)} &= (x_4, x_5) \cap M_2(1,5), \\
\sqrt{\text{saturate } ((p_0, p_1, p_2, p_3), p_5)} &= (x_4, x_5) \cap M_2(0,5).
\end{aligned}$$

4. Canonical forms and stabilizers

Recall that $G = PGL_{n+1}$ acts by the "triple action" on the projective space of matrices of linear forms; and by conjugation on the projective space of quadratic rational maps $\psi : \mathbb{P}^n \to \mathbb{P}^n$.

Theorem 4.1

For generic points $a = (a_0, ..., a_n) \in \mathbb{P}^n$, the rational maps φ_a represent a cross-section of orbits under conjugation with PGL_{n+1} on the projective space of quadratic rational maps $\psi : \mathbb{P}^n \to \mathbb{P}^n$. In other words, if $g \in PGL_{n+1}$ satisfies $[g^{-1} \circ \varphi_a \circ g] = [\varphi_b]$, then $a' = b'$ as points of $\mathbb{P}^{n-1}$, where $a' = (a_0, a_1, ..., a_{n-2}, a_n)$ and $b' = (b_0, b_1, ..., b_{n-2}, b_n)$.

Proof. We interpret assertion (4) in Theorem 3.3 as the decomposition of a generic point $x_n \neq 0$ in terms of the eigenvalues $E_i(x)$ of T_a:

$$x_n^{n-1} p_n(x) x = \sum_{i=0}^{n} p_i(x) E_i(x).$$

We pass to the affine open set $\mathbb{A}^n = \mathbb{P}^n_{x_n}$, where we set $x_n = 1$, then we interpret assertion (5) in Theorem 3.3 as a collection of "convergence" statements towards or away from the hypersurfaces $p_i(x) = 0$:

$$p_i[\varphi_a(x)] = a_i a_n^{n-i-1} x_n^{n-i} p_i(x),$$

one for each $i = 0, ..., n-2$.

The ratios $|a_i|/|a_n|$ determine the speed of each "convergence"; and must be invariant under the action of G. Indeed, if $g \in G$ is associated to the automorphism $\mathcal{L} : \mathbb{P}^n \to \mathbb{P}^n$ and the expression for $\varphi_a^g(x)$ is normalized so that every point $x = (x_0, ..., x_n)$ with $x_n = 1$ goes to a point $z = (z_0, ..., z_n)$ with $z_n = 1$, then we obtain the equation

$$q_i[\varphi_a^g(x)] = a_i a_n^{n-i-1} y_n^{n-i} q_i(x),$$

where $q_i(x) = p_i(\mathcal{L}(x))$, $y_n = (x_n \circ \mathcal{L})(x), \varphi_a^g = \mathcal{L}^{-1} \circ \varphi_a \circ \mathcal{L}$; and the expression for $\varphi_a^g(x)$ is normalized for y_n. Thus, $\varphi_a^g = \varphi_b$ implies $|a_i|/|a_n| = |b_i|/|b_n|$, for all $i = 0, ..., n-1$. $\square$

Remark 4.2 The forms $p_i(x)$ are called "entropies" in [10], because they keep track of the passage of time.

Corollary 4.3

For generic points $a = (a_0, ..., a_n) \in \mathbb{P}^n$, the matrices

$$T_a = \begin{pmatrix} a_0 x_n & 0 & \cdots & 0 & 0 \\ (a_n - a_0)x_{n-1} & a_1 x_n & \cdots & 0 & 0 \\ & \ddots & \ddots & & \\ 0 & 0 & \cdots & a_{n-1}x_n & 0 \\ 0 & 0 & \cdots & (a_n - a_{n-1})x_{n-1} & a_n x_n \end{pmatrix}$$

represent a cross-section of orbits for the triple action of PGL_{n+1} on the projective space of matrices of linear forms. In other words, if $g \in PGL_{n+1}$ satisfies $[T_a^g] = [T_b]$, then $a' = b'$ as points of $\mathbb{P}^{n-1}$, where $a' = (a_0, a_1, ..., a_{n-2}, a_n)$ and $b' = (b_0, b_1, ..., b_{n-2}, b_n)$.

Proof. Let $g = [M]$, with $M = (m_{ij}) \in GL_{n+1}$ be such that $[T_a^g] = [T_b]$ for generic points $a, b \in \mathbb{P}^n$. By Theorem 2.1, we have $[g^{-1} \circ \varphi_a \circ g] = [\varphi_b]$; and then our conclusion. $\square$

Canonical forms

After these results, we propose the matrices T_a and their associated quadratic rational maps $\varphi_a : \mathbb{P}^n \to \mathbb{P}^n$, for generic points $a \in \mathbb{P}^n$, with $a_{n-1} = 0$, as canonical forms for their orbits under the corresponding actions of $G = PGL_{n+1}$. These matrices and maps generalize and improve our previous proposals in [10].

We now proceed to study the stabilizers of our matrices and maps.

Theorem 4.4

For a generic point $a = (a_0, ..., a_n) \in \mathbb{P}^n$, the matrix T_a has stabilizer G^{T_a}, whose elements are represented by diagonal matrices diag $(d_0, ..., d_n)$ *subject to the ideal of 2×2 minors of the catalecticant matrix*

$$D = \begin{pmatrix} d_0 & d_1 & \cdots & d_{n-2} & d_{n-1} \\ d_1 & d_2 & \cdots & d_{n-1} & d_n \end{pmatrix}.$$

Thus, G^{T_a} is geometrically isomorphic to a dense open subset of the curve A; and the dimension of the orbit containing T_a is $(n+1)^2 - 2$.

Proof. Let $g = [M] \in G^{T_a}$ be represented by $M = (m_{ij}) \in GL_{n+1}$. Then $g \in G^{\varphi_a}$ and hence g fixes the curve A, the point Q and the osculating flag to A at Q. Thus, M is lower triangular.

The translate T_a^g is obtained from $[MT_aM^{-1}]$ by an invertible linear change of variables, which replaces x_n by a nonzero scalar multiple of itself and x_{n-1} by a linear combination $\alpha x_{n-1} + \beta x_n$, with $\alpha \neq 0$.

This situation forces M to be diagonal. Let us write $M = \text{diag}\ (d_0, ..., d_n)$.

A quick calculation shows that $[T_a^g]$ is represented by

$$E = \begin{pmatrix} a_0x_n & & & & 0 \\ b_{10}x_{n-1} & \ddots & & & \\ & \ddots & \ddots & & \\ & & b_{n-1,n-2}x_{n-1} & a_{n-1}x_n & \\ 0 & & & d_{n-1}^{-1}d_n(a_n - a_{n-1})x_{n-1} & a_nx_n \end{pmatrix},$$

where

$$\begin{aligned} b_{10} &= d_0^{-1}d_1(a_n - a_0), \\ b_{21} &= d_1^{-1}d_2(a_n - a_1), \\ &\cdots \\ b_{n-1,n-2} &= d_{n-2}^{-1}d_{n-1}(a_n - a_{n-2}). \end{aligned}$$

But $[T_a] = [E]$ is equivalent to the following matrix having rank one:

$$\begin{pmatrix} 1 & 1 & \cdots & 1 & 1 \\ d_0^{-1}d_1 & d_1^{-1}d_2 & \cdots & d_{n-2}^{-1}d_{n-1} & d_{n-1}^{-1}d_n \end{pmatrix},$$

which clearly amounts to rank one for the catalecticant matrix D.

Since $G = PGL_{n+1}$ has dimension $(n+1)^2 - 1$, our orbit has the stated dimension. $\square$

Theorem 4.5

For a generic point $a = (a_0, ..., a_n) \in \mathbb{P}^n$, we have the equality of stabilizers $G^{T_a} = G^{\varphi_a}$, provided $n > 2$. In that case, $\dim(G \cdot \varphi_a) = (n+1)^2 - 2$.

Proof. Let $g = [M] \in G^{\varphi_a}$, with $M = (m_{ij}) \in GL_{n+1}$. Then g fixes the curve A, the point Q and the osculating flag to A at Q. Thus, M is lower triangular.

The translate $[T_a]^g$ may be represented by a matrix S which differs from T_a by addition to any column of a multiple of a "polynomial elementary column" $E_{ij} = (0, .., 0, -x_j, 0, ..., 0, x_i, 0, ..., 0)^t$, for any given pair of indices $0 \leq i < j \leq n$, where $-x_j$ appears in the i-th position and x_i appears in the j-th position, by Proposition 2.2.

Since M is lower triangular, only the variables x_{n-1} and x_n appear in S. This restriction forces the matrix M to be diagonal, if $n > 2$. Hence, no polynomial elementary columns may alter T_a; i.e. $[T_a] = [S]$, which implies the conclusions. $\square$

Remark 4.6 In case $n = 2$, G^{φ_a} consists of all lower triangular matrices

$$C = \begin{pmatrix} c_0 & 0 & 0 \\ c_3 & c_1 & 0 \\ c_5 & c_4 & c_2 \end{pmatrix}$$

in the open subset $c_0c_1c_2 \neq 0$ of the degree eight surface given by the equations

$$c_0c_2 - c_1^2,\ c_2c_3 - 2c_1c_4,\ c_2c_5 - c_4^2.$$

In fact, if $g = [C]$, then $T_a^g = c_0c_1c_2^2T_a + (2a_1 - a_0 - a_2)c_1^2c_2c_4E_{12}^0$, where

$$E_{12}^0 = \begin{pmatrix} 0 & 0 & 0 \\ -x_2 & 0 & 0 \\ x_1 & 0 & 0 \end{pmatrix}.$$

5. Natural questions and partial answers

Several natural questions arise at this point.

(1) Given a quadratic rational map $\psi : \mathbb{P}^n \to \mathbb{P}^n$ with associated dynamical system similar to those generated by the iteration of φ_a, how far is it from some φ_a?
(2) What precisely is the orbit of each φ_a?
(3) Are there quadratic rational maps not in the orbit of any φ_a whose fixed points sets are open subsets of rational normal curves?
(4) What happens when the point a is not generic?

Generalizing [10], we say that the curve A together with the osculating flag to A at the point Q constitutes the geometry of the rational map φ_a, or else of the matrix T_a. The point of parameters $a = (a_0, ..., a_n) \in \mathbb{P}^n$ does not provide additional geometric information; as long as it is generic, it respects the geometry.

The next two examples address Question (4).

Example 5.1 The rational map $\psi : \mathbb{P}^2 \to \mathbb{P}^2$ defined by $\psi(x) = xT$ for $x = (x_0, x_1, x_2)$ and

$$T = T_{(0,b,a)} = \begin{pmatrix} 0 & 0 & 0 \\ ax_1 & bx_2 & 0 \\ 0 & (a-b)x_1 & ax_2 \end{pmatrix},$$

for $a \neq 0$, corresponds to the non-generic point of parameters $(0, b, a)$. It has the same geometry as the generic case, but the convergence to the fixed points occurs in one step, i.e. $\psi(x) \in A$, for all $x \in \mathbb{P}^2 \smallsetminus \{Q\}$, where $\{Q\} = Z(x_1, x_2)$ and $A = Z(x_1^2 - x_0x_2)$.

In [10], we had a similar dynamical system, but the conic was $Z(x_1^2 - 4x_0x_2)$, associated to the matrix

$$S = \begin{pmatrix} 0 & 0 & 0 \\ x_1 & 2x_2 & 0 \\ 0 & 2x_1 & 4x_2 \end{pmatrix}.$$

Here, $[S]^g = [T_{(0,1,2)}]$, with $g = [M]$ and $M = \operatorname{diag}(1, 2, 1)$.

If we look at the non-generic case $T = T_{(a,b,a)}$ with $a \neq 0$, we find that the line $L = Z(x_2)$ is the set of fundamental points; and that every point in $\mathbb{P}^2 \smallsetminus L$ is a fixed point.

Finally, in the non-generic case $T = T_{(a,b,0)}$ with $a \neq 0$, we find that the conic $A = Z(x_1^2 - x_0x_2)$ is the set of fundamental points; and that every point in $\mathbb{P}^2 \smallsetminus A$ goes to the fundamental point $Q = (1, 0, 0)$.

EXAMPLE 5.2 The rational map $\varphi : \mathbb{P}^3 \to \mathbb{P}^3$ defined by $\varphi(x) = xT$ for $T = T_{(0,0,b,a)}$, with $a \neq 0$, has the line $L = Z(x_2, x_3)$ as the set of fundamental points; and the standard twisted cubic curve A as the closure of the fixed-point set. Here, $\varphi(x) \subseteq Z(x_2^2 - x_1x_3)$, for all $x \in \mathbb{P}^3 \setminus L$, but it is not the case that $\varphi(x) \subseteq A$, for these points. However, $\varphi^2(x) \subseteq A$, for all $x \in \mathbb{P}^3 \setminus L$.

In the non-generic case $T = T_{(b,0,c,a)}$, with $ab(a-b) \neq 0$, we find the line $L = Z(x_2, x_3)$ as the set of fundamental points; and the standard twisted cubic curve A as the closure of the fixed-point set. Here, $\varphi(x) \subseteq Z(x_2^2 - x_1x_3)$, for all $x \in \mathbb{P}^3 \setminus L$.

The next example compares the present canonical forms with our previous proposal in [10].

EXAMPLE 5.3 The general nonsingular conic case in [10] was constructed with a matrix M and its associated map $\psi(x) = xM$. Here, the triangular matrix B satisfies $[\psi] = [\mathcal{L}^{-1} \circ \varphi_{(a,2a+b,2a+2b)} \circ \mathcal{L}]$, where $\mathcal{L}(x) = xB$, $\varphi_{(a,2a+b,2a+2b)}$ is a canonical quadratic rational map; and

$$M = \begin{pmatrix} ax_2 & 0 & 0 \\ \frac{1}{2}bx_1 + \frac{1}{2}ax_2 & (2a+b)x_2 & 0 \\ 0 & bx_1 & 2(a+b)x_2 \end{pmatrix}, \; B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ c & d & t \end{pmatrix},$$

with

$$t = \frac{2a+4b}{b}, \; d = \frac{at}{4a+8b}, \; c = \frac{d^2}{t}.$$

The next result addresses Questions (1-2).

Theorem 5.4

Let A be the smooth conic $x_1^2 = x_0x_2$; and let $B = (1,0,0)$. Assume that $\psi = (\psi_0, \psi_1, \psi_2) : \mathbb{P}^2 \to \mathbb{P}^2$ is a quadratic rational map such that

(1) *The algebraic set defined by the ideal (ψ_0, ψ_1, ψ_2) has support $\{B\}$.*
(2) *The algebraic set defined by the determinantal ideal I of all 2×2 minors from the 2×3 matrix with rows $x = (x_0, x_1, x_2)$ and $\psi(x)$ has support A.*
(3) *The only possible multiple point in $Z(I)$ is B.*

Then $\psi = \varphi_{a,b}$, with $\varphi_{a,b}(x) = xT_{a,b}$, for adequate parameters $(a,b) \in \mathbb{P}^1$, where

$$T_{a,b} = T_{(a,0,b)} = \begin{pmatrix} ax_2 & 0 & 0 \\ (b-a)x_1 & 0 & 0 \\ 0 & bx_1 & bx_2 \end{pmatrix}.$$

Proof. Let $y_1, ..., y_{18}$ be parameters, in order to write a general quadratic rational map as follows:

$$\psi = (y_1x_0^2 + y_2x_1^2 + y_3x_2^2 + y_4x_0x_1 + y_5x_0x_2 + y_6x_1x_2, ...).$$

Requiring the conic A to be contained in the zero set of I, we reduce the above expression for ψ to the following one, with six parameters:

$$\psi \;\; = \;\; (ax_1^2 + bx_0x_1 + cx_0^2 + dx_0x_2 - dx_1^2,$$

$$\begin{aligned} &ax_1x_2 + bx_1^2 + cx_0x_1 + fx_0x_2 - fx_1^2, \\ &ax_2^2 + bx_1x_2 + cx_0x_2 - ex_0x_2 + ex_1^2). \end{aligned} \tag{5.1}$$

Requiring B to satisfy the ideal $J = (\psi_0, \psi_1, \psi_2)$, we obtain $c = 0$. Requiring the ideal I to be supported by the conic A, with B as unique possible multiple point, yields $e = f = 0$ and $a \neq d$.

Requiring B to be the unique possible multiple point in $Z(J)$ implies $b = 0$, but then $\psi = \varphi_{(d,0,a)}$, with $a \neq d$. □

Consider Question (3) for the simple case of the plane. A natural candidate for fundamental point set for a rational map ψ with fixed point set equal to a dense open subset of A, is the line $L = Z(x_2)$, tangent to A at B. We have the following negative answer.

Corollary 5.5

Let A be the conic $x_1^2 = x_0x_2$. Assume that $\psi : \mathbb{P}^2 \to \mathbb{P}^2$ is a quadratic rational map with fixed point set equal to a dense open subset of A. Then, the set of fundamental points of ψ cannot be the line $L = Z(x_2)$, tangent to A at B.

Proof. We start with a quadratic rational map $\psi : \mathbb{P}^2 \to \mathbb{P}^2$ with six parameters such that A satisfies the ideal I, as in the Theorem. Then, we saturate the ideal I with respect to the conic A, to obtain the condition $ac - cd + de - bf + f^2 = 0$; and the linear equations

$$\begin{aligned} &ex_1 + fx_2, fx_0 + ax_1 - dx_1, \\ &ex_0 - ax_2 + dx_2, cx_0 + bx_1 - fx_1 + dx_2. \end{aligned}$$

If this system reduces to the equation $x_2 = 0$, then $b = c = e = f = 0$ and $a = d \neq 0$; and the map must be $\psi(x) = (x_0x_2, x_1x_2, x_2^2)$, which is a contradiction. □

We continue to address Questions (1-2), for the twisted cubic case. Here, we have a point of parameters $a = (a_0, a_1, a_2, a_3) \in \mathbb{P}^3$, a matrix of linear forms T_a and a rational map $\varphi_a : \mathbb{P}^3 \to \mathbb{P}^3$.

Proposition 5.6

Let $\varphi = \varphi_a$ be a quadratic rational map of $\mathbb{P}^3$, with a generic point of parameters a, let I be the determinantal ideal of all 2×2 minors of the 2×4 matrix with rows x and $\varphi(x)$; and let K be the ideal generated by $\{\varphi_0, ..., \varphi_3\}$. Then

- *We have the primary decomposition $I = \mathfrak{p} \cap \mathfrak{q}$, with $\mathfrak{p}$ a prime ideal, corresponding to the standard twisted cubic curve A; and $\mathfrak{q}$ minimally generated by $\{x_3^2, x_2x_3, x_2^2, q\}$, where q is a nonzero quadratic form contained in the ideal (x_2, x_3).*
- *K is minimally generated by a set $\{x_3^2, x_2x_3, h_1, h_2\}$, where h_1 and h_2 are nonzero quadratic forms contained in the ideal (x_2, x_3).*

Proof. This is a simple calculation with Macaulay2. □

We are ready to use these geometric ingredients together with our previous entropies, see [10], to obtain the following result.

Theorem 5.7

Let A be the standard twisted cubic curve, image of the Veronese map $\nu : \mathbb{P}^1 \to \mathbb{P}^3$, given by $\nu(t,u) = (t^3, t^2u, tu^2, u^3)$, let $Q = (1,0,0,0) \in A$; and let $L = Z(x_2, x_3)$, the tangent line to A at Q. Assume that $\psi = (\psi_0, ..., \psi_3) : \mathbb{P}^3 \to \mathbb{P}^3$ is a quadratic rational map such that

(1) *The algebraic set defined by the determinantal ideal I of all 2×2 minors from the 2×4 matrix with rows $x = (x_0, ..., x_3)$ and $\psi(x)$, has support $A \cup L$.*
(2) *We have the primary decomposition $I = \mathfrak{p} \cap \mathfrak{q}$, with $\mathfrak{p} = I(A)$ prime, generated by $\{x_1x_3 - x_2^2, x_0x_3 - x_1x_2, x_0x_2 - x_1^2\}$; and $\mathfrak{q}$ minimally generated by $\{x_3^2, x_2x_3, x_2^2, q\}$, where q is a nonzero quadratic form contained in the ideal (x_2, x_3). Thus, $\sqrt{\mathfrak{q}} = (x_2, x_3)$ and $Z(\mathfrak{q}) = L$.*
(3) *The ideal K generated by $\{\psi_0, \psi_1, \psi_2, \psi_3\}$, is minimally generated by the set $\{x_3^2, x_2x_3, h_1, h_2\}$, where h_1 and h_2 are nonzero quadratic forms contained in the ideal (x_2, x_3). Thus, $Z(K) = L$.*
(4) *There is a quadratic entropy $p(x) \in \mathfrak{p}$ satisfying a relation of the form*

$$p(\psi(x)) = \alpha x_3^2 p(x),$$

for some nonzero scalar α.
(5) *There is a cubic entropy $r(x) \in \mathfrak{p}$ satisfying two unique relations of the form*

$$r(\psi(x)) = \beta x_3^3 r(x),$$

for some nonzero scalar β; and

$$c_0 x_3 \psi(x) = c_1 x_3^2 x + c_2 p(x) B + c_3 r(x) Q,$$

for scalars c_0, c_1, c_2, c_3; $p(x)$ the previous quadratic entropy; and B an adequate linear parametrization of L.

Then $\psi(x) = xS$, for the matrix of linear forms

$$S = \begin{pmatrix} bx_3 & 0 & 0 & 0 \\ cx_2 - ax_3 & (b+c-d)x_3 & 0 & 0 \\ ax_2 & dx_2 & 0 & 0 \\ 0 & 0 & (b+c)x_2 & (b+c)x_3 \end{pmatrix}, \tag{5.2}$$

for adequate parameters a, b, c, d.

Proof. We consider forty parameters $y_1, ..., y_{40}$, in order to write

$$\psi = (y_1x_0^2 + \cdots + y_4x_3^2 + y_5x_0x_1 + \cdots + y_{10}x_2x_3, ...).$$

The condition $Z(K) \supseteq L$ is equivalent to

$$\begin{aligned} y_1 = y_2 = y_5 = y_{11} = y_{12} = y_{15} = 0, \\ y_{21} = y_{22} = y_{25} = y_{31} = y_{32} = y_{35} = 0. \end{aligned} \tag{5.3}$$

The condition $Z(I) \supseteq A$ is then equivalent to

$$y_4 \;=\; y_{10} = y_{14} = y_{16} = y_{20} = y_{24} = y_{26} = y_{36} = 0,$$

$$\begin{aligned} y_{28} &= -y_{27},\ y_{38} = -y_{37},\ y_{34} = y_{30}, \\ y_9 &= -y_3,\ y_{30} = y_7 + y_8,\ y_{39} = -y_{33},\ y_{40} = y_6, \\ y_{18} &= y_6 - y_{17},\ y_{29} = y_6 - y_{23},\ y_{19} = y_7 + y_8 - y_{13}. \end{aligned} \tag{5.4}$$

Since the ideal $K + I(A)$ admits the basis

$$\begin{aligned} x_1x_3 - x_2^2,\ x_0x_3 - x_1x_2,\ x_0x_2 - x_1^2, \\ (x_2y_6 + x_3y_7 + x_3y_8)x_3, \\ (x_1y_6 + x_2y_7 + x_2y_8)x_3, \\ (x_0y_6 + x_1y_7 + x_1y_8)x_3, \\ x_0(x_2y_6 + x_3y_7 + x_3y_8); \end{aligned} \tag{5.5}$$

we see that Condition (3) implies $y_6 = 0$, because otherwise we could find a point $x = (x_0, x_1, x_2, x_3) \in A$, with $x_3 \neq 0$, satisfying the above basis.

In order to use effectively Condition (2), we saturate the ideal I with respect to the twisted cubic $(x_1x_3 - x_2^2, x_0x_3 - x_1x_2, x_0x_2 - x_1^2)$; and call J the result. Then we write $J' = J + (x_3^2, x_2x_3, x_2^2)$. The ideal J' is minimally generated by $\{x_3^2, x_2x_3, x_2^2\}$ together with all of the following polynomials

$$\begin{aligned} &x_1x_3(y_{27}y_{33} - y_{23}y_{37}), \\ &x_1x_2(y_{27}y_{33} - y_{23}y_{37}), \\ &x_0x_3(y_{27}y_{33} - y_{23}y_{37}), \\ &x_0x_2(y_{27}y_{33} - y_{23}y_{37}), \\ &x_3(x_1y_8y_{23} + x_1y_3y_{27} + x_0y_{17}y_{23} - x_0y_{13}y_{27}), \\ &x_3(x_1y_8y_{33} + x_1y_3y_{37} + x_0y_{17}y_{33} - x_0y_{13}y_{37}), \\ &x_2(x_1y_8y_{33} + x_1y_3y_{37} + x_0y_{17}y_{33} - x_0y_{13}y_{37}), \\ &x_2(x_1y_8y_{23} + x_1y_3y_{27} + x_0y_{17}y_{23} - x_0y_{13}y_{27}), \\ &x_1(x_1x_3y_{33} + x_1x_2y_{37} - x_0x_3y_{37}), \\ &x_1(x_1x_3y_{23} + x_1x_2y_{27} - x_0x_3y_{27}), \\ &x_0(x_1x_3y_{33} + x_1x_2y_{37} - x_0x_3y_{37}), \\ &x_0(x_1x_3y_{23} + x_1x_2y_{27} - x_0x_3y_{27}), \\ &(x_1^2x_3y_3 - x_1^2x_2y_8 + x_0x_1x_3y_8 - x_0x_1x_3y_{13} - x_0x_1x_2y_{17} + x_0^2x_3y_{17}). \end{aligned}$$

The ideal J' is our candidate for $\mathfrak{q}$. We obtain some necessary conditions on the parameters y_i in order to guarantee that J' is a primary ideal satisfying Condition (2): The ideal $\mathfrak{a}$ must be satisfied, since otherwise there would be too many minimal generators for J', where $\mathfrak{a}$ is the determinantal ideal $\mathfrak{b}$ generated by all 2×2 minors from

$$\begin{pmatrix} y_3 & y_{13} & y_{23} & y_{33} \\ -y_8 & y_{17} & y_{27} & y_{37} \end{pmatrix}$$

with $y_3y_{17} + y_8y_{13}$ missing. We have the primary decomposition

$$\mathfrak{a} = (y_{23}, y_{27}, y_{33}, y_{37}) \cap \mathfrak{b}.$$

If we assume that our remaining parameters satisfy the determinantal ideal $\mathfrak{b}$, i.e. the missing determinant $y_3y_{17} + y_8y_{13}$, we contradict Condition (2). Thus,

$$y_{23} = y_{27} = y_{33} = y_{37} = 0,\ y_3y_{17} + y_8y_{13} \neq 0,$$

because $y_{23}, y_{27}, y_{33}, y_{37} \in saturate(\mathfrak{a}, y_3y_{17} + y_8y_{13})$.

At this point, the primary decomposition of the ideal I satisfies Condition (2). Furthermore, this is the earliest stage at which $\psi(x) - x \in L$, for all $x \in \mathbb{P}^3 \smallsetminus L$. This is also as far as we can stretch the geometric requirements on ψ.

We proceed to use Condition (4). Let $f(x) = x_2^2 - x_1x_3$, $g(x) = x_1x_2 - x_0x_3$, and $h(x) = x_1^2 - x_0x_2$; and let $F(x) = f(\psi(x))$, $G(x) = g(\psi(x))$ and $H(x) = h(\psi(x))$.

The quadratic entropy $p(x)$ must be a scalar linear combination of f, g and h. But H contains the term $-y_{17}x_1^2x_2^2$, which does not appear neither in F nor in G; and cannot be canceled in any candidate for an entropy where h appears. Similarly, G contains the term $-y_{17}x_1x_2^2x_3$, not appearing in F, leaving $y_{17} = 0$ as the only possibility for an entropy p to exist. This is achieved with $p = f$.

We are left with four parameters, which we relabel as follows:

$$y_3 = a,\ y_7 = b,\ y_8 = c,\ y_{13} = d.$$

Then the entropic constant, see [10], corresponding to f above is

$$\frac{b+c-d}{b+c}.$$

Condition (2) now implies $cd \neq 0$. Thus, the new parameters must satisfy the restriction $cd(b+c)(b+c-d) \neq 0$.

We proceed to use Condition (5). We immediately obtain

$$\begin{aligned} B &= (cx_2 - ax_3, (c-d)x_3, 0, 0), \\ r &= ax_1x_3^2 - ax_2^2x_3 + cx_0x_3^2 - cx_1x_2x_3 - dx_0x_3^2 + dx_2^3. \end{aligned}$$

We have obtained the cubic entropy r with corresponding entropic constant

$$\frac{b}{b+c}.$$

The parametrization B above is the only possible one satisfying Condition (5). We can now verify directly that our map ψ satisfies $\psi(x) = xS$, for the stated matrix of linear forms S. $\square$

Remark 5.8 This result tells us that perhaps our canonical form is missing one parameter. The next result shows that this is not the case.

Proposition 5.9

Given parameters a, b, c, d; let $\psi_{(a,b,c,d)} : \mathbb{P}^3 \to \mathbb{P}^3$ be the rational map defined as $\psi_{(a,b,c,d)}(x) = xS_{(a,b,c,d)}$, for $x \in \mathbb{P}^3$; and

$$S_{(a,b,c,d)} = \begin{pmatrix} bx_3 & 0 & 0 & 0 \\ cx_2 - ax_3 & (b+c-d)x_3 & 0 & 0 \\ ax_2 & dx_2 & 0 & 0 \\ 0 & 0 & (b+c)x_2 & (b+c)x_3 \end{pmatrix}.$$

Let A, L, Q be as in the preceding theorem. For generic values of the parameters b, c, d, we have

(1) *The set of fundamental points of $\psi_{(a,b,c,d)}$ is the line $L = Z(x_2, x_3)$.*
(2) *The set of fixed points of $\psi_{(a,b,c,d)}$ is $A \setminus Q$.*
(3) *If there exists $g \in G$ with $\psi^g_{(a,b,c,d)} = \psi_{(t,u,v,w)}$, then $(b,c,d) = (u,v,w)$ as points in $\mathbb{P}^2$.*
(4) *The transporter $Trans_G(\psi_{(a,b,c,d)}, \psi_{(t,b,c,d)})$ is geometrically isomorphic to an open set of the twisted cubic curve. In particular, it is nonempty.*
(5) *A complete set of invariants for the orbits containing maps $\psi_{(a,b,c,d)}$ under the action of PGL_4, is the point $(b,c,d) \in \mathbb{P}^2$.*

Proof. Properties (1) and (2) can be verified easily. In fact, they were imposed on all $\psi_{(a,b,c,d)}$.

Given two generic points of parameters (b,c,d) and (u,v,w), any element $g \in Trans_G(\varphi_{(a,b,c,d)}, \varphi_{(t,u,v,w)})$ must preserve the flag

$$Q \subset L \subset H = Z(x_3).$$

Thus, g may be represented by a lower triangular matrix

$$M = \begin{pmatrix} y_0 & 0 & 0 & 0 \\ y_1 & y_2 & 0 & 0 \\ y_3 & y_4 & y_5 & 0 \\ y_6 & y_7 & y_8 & y_9 \end{pmatrix}.$$

We compute the composition $g \circ \varphi_{(a,b,c,d)} \circ g^{-1}$, which we identify with a 4×1 vector $\varphi^g_{(a,b,c,d)}$ of quadratic forms. We then consider $\varphi_{(t,u,v,w)}$ as a 4×1 vector of quadratic forms; and then we look at the ideal I of 2×2 minors from the matrix with rows $\varphi^g_{a,b,c,d}$ and $\varphi_{(t,u,v,w)}$.

The ideal J generated by the coefficients with respect to the variables x_i of the elements in I establishes the conditions for M to represent an element in $Trans_G(\varphi_{(a,b,c,d)}, \varphi_{(t,u,v,w)})$ and the conditions for $\varphi_{(t,u,v,w)}$ to be in the orbit of $\varphi_{(a,b,c,d)}$.

(3) In the general case, we require

$$bcduvw(b+c)(u+v)(b+c-d)(v-w)(u+v-w)y_0y_2y_5y_9 \neq 0,$$

before we eliminate the variables y_i, to obtain the ideal of 2×2 minors from the matrix

$$\begin{pmatrix} b & c & d \\ u & v & w \end{pmatrix}.$$

(4) In case $(b,c,d) = (u,v,w)$, we obtain that the diagonal entries of M describe an open part of a twisted cubic as equivalent to

$$g \in Trans_G(\varphi_{(a,b,c,d)}, \varphi_{(t,b,c,d)}).$$

A collection of linear equations determine the remaining entries of M.

(5) follows immediately from (3) and (4). $\square$

Question (3) admits a positive answer in the case of the rational normal quartic A that appeared in [10]. The rational maps studied there, have as fixed point set $A \smallsetminus Q$, with $Q = (1, 0, 0, 0, 0)$, the same as in the present case. But, the set of fundamental points in [10] is the tangent line $Z(x_2, x_3, x_4)$ to A at Q, different from the tangent plane $Z(x_3, x_4)$ obtained presently.

6. Biological discussion

The dynamical systems presently studied are mathematical generalizations of the simplest cases. The iteration of $\varphi_{(0,0,1)} : \mathbb{P}^2 \to \mathbb{P}^2$ describes the Hardy-Weinberg Theorem in Population Genetics, after change of basis. This states that focusing our attention at a locus with two alleles, any population evolving via outcrossing, with no mutations, no selfing and no selection parameters, reaches equilibrium in the next generation. The equilibrium genotype frequencies are described by the smooth conic.

For the same population, we may allow selfing and inbreeding depression, in which case we obtain a dynamical system associated to a generic map $\varphi_\alpha : \mathbb{P}^2 \to \mathbb{P}^2$.

The final example in the previous section, describes a population at a locus with two alleles, whose individuals carry quadruple genetic information instead of the common double information. This phenomenon is called tetraploidy, see [13]. The geometry for that dynamical system involves the rational normal quartic. It is a slight variant of the geometry obtained from our present general systems.

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