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Local Riesz transforms characterization of local Hardy spaces

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ABSTRACT

For $0 < p \leq 1$, let $h^p(\mathbf{R}^n)$ denote the local Hardy space. Let $\hat{\theta}$ be a smooth, compactly supported function, which is identically one in a neighborhood of the origin. For $k = 1, \dots, n$, let $(r_k f)^\wedge(\xi) = -i(1 - \hat{\theta}(\xi))\xi_k/|\xi|\hat{f}(\xi)$ be the local Riesz transform and define $(r_0 f)^\wedge(\xi) = (1 - \hat{\theta}(\xi))\hat{f}(\xi)$. Let Ψ be a fixed Schwartz function with $\int \Psi dx = 1$, let $M > 0$ be an integer and suppose $(n-1)/(n+M-1) < p \leq 1$.

We show that a tempered distribution f which is restricted at infinity belongs to $h^p(\mathbf{R}^n)$ if and only if $\theta * f \in h^p(\mathbf{R}^n)$ and there exists a constant $A > 0$ such that for all ε with $0 < \varepsilon \leq 1$ we have $\sum_{M \leq |\alpha| \leq M+1} \|r^\alpha(f) * \Psi_\varepsilon\|_{L^p(\mathbf{R}^n)} \leq A$. Here, $\Psi_\varepsilon(x) = \varepsilon^{-n}\Psi(x/\varepsilon)$, $\alpha = (\alpha_0, \dots, \alpha_n) \in \mathbf{N}^{n+1}$, r^α , as usual, denotes the composition $r_0^{\alpha_0} \circ \dots \circ r_n^{\alpha_n}$. This result extends to the local Hardy spaces the analogous characterization of the classical Hardy spaces $H^p(\mathbf{R}^n)$ (see e.g. [9, Chapter III.5.16]).

1. Introduction

The local Hardy spaces $h^p(\mathbf{R}^n)$ were introduced and studied by Goldberg in [2], see also [9, Chapter III.5]. They are spaces of tempered distributions that are much better suited to problems associated with partial differential equations than the classical Hardy spaces $H^p(\mathbf{R}^n)$. We recall that a tempered distribution f belongs to $h^p(\mathbf{R}^n)$ if, for some Schwartz function Φ with $\int \Phi dx \neq 0$, the maximal function $M_\Phi f(x) = \sup_{0 < t \leq 1} |(f * \Phi_t)(x)|$ is in $L^p(\mathbf{R}^n)$, where $\Phi_t(x) = t^{-n}\Phi(x/t)$. For $0 < p \leq 1$,

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an equivalent way of looking at $h^p(\mathbf{R}^n)$ is given by the atomic decomposition, which expresses elements in $h^p(\mathbf{R}^n)$ as sums of very simple elements called atoms. We defer to Section 2 for more details.

In this paper we give another characterization of $h^p(\mathbf{R}^n)$ for $0 < p \leq 1$ in terms of appropriate singular integrals in a way that is analogous to the characterization of the classical Hardy spaces by the Riesz transforms (see e.g. [9, Chapter III.5.16]).

We will denote the Fourier transform of a tempered distribution f by $\hat{f}$. If f is integrable

$$\hat{f}(\xi) = \int_{\mathbf{R}^n} f(x) e^{-2\pi i \xi \cdot x} dx.$$

For $k = 1, \dots, n$ let

$$(R_k f)^\wedge(\xi) = -i \frac{\xi_k}{|\xi|} \hat{f}(\xi)$$

be the classical Riesz transform.

We say that a tempered distribution f is restricted at infinity if, for all sufficiently large $q < \infty$, $f * \Phi \in L^q(\mathbf{R}^n)$ for all Schwartz functions Φ .

A classical result concerning the standard H^p spaces says that if Ψ is a fixed Schwartz function with $\int \Psi dx = 1$, $M > 0$ is an integer such that $(n-1)/(n+M-1) < p \leq 1$, then a tempered distribution f that is restricted at infinity belongs to $H^p(\mathbf{R}^n)$ if and only if there exists a constant $A > 0$ such that for all $\varepsilon > 0$ we have $\sum_{|\beta| \leq M} \|R^\beta(f) * \Psi_\varepsilon\|_{L^p(\mathbf{R}^n)} \leq A$. Here $\beta = (\beta_1, \dots, \beta_n)$ is an n -tuple of natural numbers, R^β , as usual, denotes the composition $R_1^{\beta_1} \circ \dots \circ R_n^{\beta_n}$. The proof of this result is closely connected to the characterization of $H^p(\mathbf{R}^n)$ in terms of systems of higher conjugate harmonic functions as described in [10, 1].

In this paper we extend to the local Hardy spaces $h^p(\mathbf{R}^n)$, $0 < p \leq 1$, the above characterization of $H^p(\mathbf{R}^n)$ spaces.

Let $\hat{\theta}$ be a smooth compactly supported function which is identically one in a neighborhood of the origin. For $k = 1, \dots, n$ let

$$(r_k f)^\wedge(\xi) = -i(1 - \hat{\theta}(\xi)) \xi_k / |\xi| \hat{f}(\xi) \quad (1)$$

be the local Riesz transform and define

$$(r_0 f)^\wedge(\xi) = (1 - \hat{\theta}(\xi)) \hat{f}(\xi). \quad (2)$$

Our main result shows that if Ψ is a fixed Schwartz function such that $\int \Psi dx = 1$, $M > 0$ is an integer such that $(n-1)/(n+M-1) < p \leq 1$, then a tempered distribution f which is restricted at infinity belongs to $h^p(\mathbf{R}^n)$ if and only if $\theta * f \in h^p(\mathbf{R}^n)$ and there exists a constant $A > 0$ such that for all ε with $0 < \varepsilon \leq 1$ we have $\sum_{M \leq |\alpha| \leq M+1} \|r^\alpha(f) * \Psi_\varepsilon\|_{L^p(\mathbf{R}^n)} \leq A$. Here $\alpha = (\alpha_0, \dots, \alpha_n)$ is a $n+1$ -tuple of natural numbers, $r^\alpha = r_0^{\alpha_0} \circ \dots \circ r_n^{\alpha_n}$.

In [2] Goldberg states that this result holds true in the case $1 - 1/n < p \leq 1$; the proof of this fact is closely related to the links between the elements of $h^p(\mathbf{R}^n)$ and the systems of conjugate harmonic functions and can be obtained by suitably modifying the arguments of [8, page 221], (see also [9, Chapter III.4.3]) for the analogous characterization of the classical Hardy spaces. Thus, we extend this property to the general case $0 < p \leq 1$.

An application of this result appears in [7], where it was used to show that a classical Fourier integral operator of order $\mu \leq -\frac{n-1}{2}(\frac{1}{p} - \frac{1}{2})$ is bounded on $h^p(\mathbf{R}^n)$.

In [3] Miyachi pointed out that for translation invariant operators the $H^p \rightarrow L^p$ boundedness is equivalent to the $H^p \rightarrow H^p$ boundedness; this uses the H^p boundedness of the classical Riesz-transforms. For related results in the H^p -setting see also [4, 5, 6]. In particular, in [3] the boundedness of some Fourier multipliers on the classical Hardy spaces H^p is obtained by exploiting the characterization of H^p by means of the Riesz transforms. Thus, in this paper we extend Miyachi's approach to the local h^p spaces, by generalizing the local Riesz transforms characterization of the h^p spaces via the transforms defined in (1).

The paper is organized as follows. We begin by recalling the definition of the local Hardy spaces and some of their basic properties. The results in this section are somewhat elementary, but we include some of their proofs since they do not appear explicitly in the literature. In Section 3 we establish some properties of $h^p(\mathbf{R}^n)$ spaces related to systems of conjugate harmonic functions. Finally, in Section 4 we prove the characterization of the local Hardy spaces in terms of the local Riesz transforms. We believe that such a criterion could be quite useful to prove that a given (bounded) tempered distribution f belongs to $h^p(\mathbf{R}^n)$, since pointwise estimates for the maximal function $M_\Phi f$ are often difficult to obtain.

In the following the symbol C will denote a constant which may vary from one formula to the next one.

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2. A few basic properties of local Hardy spaces

For $p > 0$ the local Hardy spaces $h^p(\mathbf{R}^n)$ can be described as follows. Let $\mathcal{S}(\mathbf{R}^n)$ denote the Schwartz space on which we have a countable family of seminorms $\|\cdot\|_{\alpha,\beta}$ given by

$$\|\Phi\|_{\alpha,\beta} = \sup_{x \in \mathbf{R}^n} |x^\alpha \partial_x^\beta \Phi(x)|, \quad \Phi \in \mathcal{S}(\mathbf{R}^n),$$

where, as usual, we write

$$x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}, \quad \partial_x^\beta = \frac{\partial^{\beta_1}}{\partial x_1^{\beta_1}} \cdots \frac{\partial^{\beta_n}}{\partial x_n^{\beta_n}}$$

for $\alpha = (\alpha_1, \dots, \alpha_n)$ and $\beta = (\beta_1, \dots, \beta_n) \in \mathbf{N}^n$. We also set $|\alpha| = \alpha_1 + \cdots + \alpha_n$. We will denote by $\mathcal{S}'(\mathbf{R}^n)$ the space of tempered distributions.

For any $\Phi \in \mathcal{S}(\mathbf{R}^n)$ and any $f \in \mathcal{S}'(\mathbf{R}^n)$, let $M_\Phi f$ be the maximal function

$$M_\Phi f(x) = \sup_{0 < t \leq 1} |(f * \Phi_t)(x)|, \quad (3)$$

where $\Phi_t(x) = t^{-n} \Phi(x/t)$.

Let Λ be a finite set of multi-indices of $\mathbf{N}^n$ and let $\mathcal{F} = \{\|\cdot\|_{\alpha,\beta} : \alpha, \beta \in \Lambda\}$ be any finite collection of seminorms on $\mathcal{S}(\mathbf{R}^n)$, we set

$$\mathcal{S}_\mathcal{F} = \{\Phi \in \mathcal{S}(\mathbf{R}^n) : \|\Phi\|_{\alpha,\beta} \leq 1 \text{ for all } \|\cdot\|_{\alpha,\beta} \in \mathcal{F}\} \quad (4)$$

and we define the grand maximal function $\mathcal{M}_{\mathcal{F}}f$ by

$$\mathcal{M}_{\mathcal{F}}f(x) = \sup_{\Phi \in \mathcal{F}} M_{\Phi}f(x). \quad (5)$$

Here, and in the following, we will denote by (x, t) the coordinates of $\mathbf{R}^{n+1}$, where $x = (x_1, \dots, x_n)$.

Let $S = \{(x, t) \in \mathbf{R}^{n+1} : 0 < t < 1\}$ be the strip in $\mathbf{R}^{n+1}$ and let $P_t^0(x)$, $P_t^1(x) = P_{1-t}^0(x)$ and $P_t(x) = P_t^0(x) + P_t^1(x)$ be the Poisson kernels for S whose Fourier transforms in the x -variable are

$$\widehat{P}_t^0(\xi) = \frac{\sinh((1-t)2\pi|\xi|)}{\sinh(2\pi|\xi|)}, \quad \widehat{P}_t^1(\xi) = \frac{\sinh(2\pi|\xi|t)}{\sinh(2\pi|\xi|)}, \quad \widehat{P}_t(\xi) = \frac{\cosh((1-2t)\pi|\xi|)}{\cosh(\pi|\xi|)}, \quad (6)$$

respectively. Note that $P_t^0(x) \rightarrow \delta_0$ as $t \rightarrow 0^+$ and $P_t^0(x) \rightarrow 0$ as $t \rightarrow 1^-$ while $P_t(x)$ is symmetric with respect to $t = 1/2$. Moreover, all the Poisson kernels belong to $\mathcal{S}(\mathbf{R}^n)$.

Finally, for any $f \in \mathcal{S}'(\mathbf{R}^n)$, let

$$u(x, t) = (f * P_t)(x)$$

be the Poisson integral of f , and let

$$u^*(x) = \sup_{|x-y| \leq t < \frac{1}{2}} |u(y, t)|$$

denote the nontangential maximal function of u .

We may state the following theorem whose proof can be found in [2].

Theorem 2.1

For $f \in \mathcal{S}'(\mathbf{R}^n)$ and $0 < p \leq \infty$, the following conditions are equivalent:

- (i) there is a $\Phi \in \mathcal{S}(\mathbf{R}^n)$ with $\int \Phi dx \neq 0$ such that $M_{\Phi}f \in L^p(\mathbf{R}^n)$;
- (ii) there is a collection $\mathcal{F}$ such that $\mathcal{M}_{\mathcal{F}}f \in L^p(\mathbf{R}^n)$;
- (iii) $u^* \in L^p(\mathbf{R}^n)$.

If any of these three maximal equivalent properties holds, then we say that f belongs to $h^p(\mathbf{R}^n)$. It can also be proved that the finite set of seminorms $\mathcal{F}$ appearing in (ii) can be chosen independent of f , and to depend only on p and n . The quantities

$$\|M_{\Phi}f\|_{L^p}, \quad \|\mathcal{M}_{\mathcal{F}}f\|_{L^p}, \quad \|u^*\|_{L^p}$$

are mutually comparable, with bounds independent of f . Any of these can be taken to be the “norm” on the space $h^p(\mathbf{R}^n)$ and denoted by $\|f\|_{h^p(\mathbf{R}^n)}$.

For all $p > 0$ the classical Hardy space $H^p(\mathbf{R}^n)$ is contained properly in $h^p(\mathbf{R}^n)$ and for $1 < p \leq \infty$ the local Hardy space $h^p(\mathbf{R}^n)$ may be identified with $L^p(\mathbf{R}^n)$.

The next lemma whose proof can be found in [2], is a very useful tool in passing from $H^p(\mathbf{R}^n)$ to $h^p(\mathbf{R}^n)$.

Lemma 2.2

Let $f \in h^p(\mathbf{R}^n)$ with $0 < p \leq 1$ and let $\Phi \in \mathcal{S}(\mathbf{R}^n)$ be a function satisfying the two following conditions

$$\int_{\mathbf{R}^n} \Phi(x) dx = 0 \quad \text{and} \quad \int_{\mathbf{R}^n} x^\alpha \Phi(x) dx = 0$$

for all $\alpha \in \mathbf{N}^n$ with $0 < |\alpha| \leq [n(1/p - 1)]$. Then there exists a positive constant C such that

$$\|f - f * \Phi\|_{H^p(\mathbf{R}^n)} \leq C \|f\|_{h^p(\mathbf{R}^n)},$$

so in particular $f - f * \Phi \in H^p(\mathbf{R}^n)$.

Here $[x]$ denotes the integral part of the real number x .

When $0 < p \leq 1$, each tempered distribution $f \in h^p(\mathbf{R}^n)$ admits an atomic decomposition

$$f = \sum_j \lambda_j a_j,$$

where λ_j are constants such that $\sum_j |\lambda_j|^p \approx \|f\|_{h^p(\mathbf{R}^n)}^p$ and the a_j are p -atoms. If we denote by $B(x_0, \delta)$ and $|B(x_0, \delta)|$ the ball centered at $x_0 \in \mathbf{R}^n$ of radius $\delta > 0$ and its Lebesgue measure, respectively, a function a is called a p -atom associated to the ball $B(x_0, \delta)$ if:

- (i) a is supported in $B(x_0, \delta)$;
- (ii) $|a(x)| \leq |B(x_0, \delta)|^{-1/p}$ a.e.;

and

- (iii) $\int_{\mathbf{R}^n} x^\alpha a(x) dx = 0$ if $\delta \leq 1$ and for all $\alpha \in \mathbf{N}^n$ with $|\alpha| \leq 1 + [n(1/p - 1)]$ and no further condition is required if $\delta > 1$.

A number of the standard properties of the classical Hardy spaces are also valid for $h^p(\mathbf{R}^n)$ spaces in suitably modified form.

Let $\Phi \in \mathcal{S}(\mathbf{R}^n)$ with $\int \Phi dx \neq 0$ and let $M_\Phi^* f$ be the nontangential maximal function of f defined by

$$M_\Phi^* f(x) = \sup_{|y-x| < t \leq 1} |(f * \Phi_t)(y)|. \quad (7)$$

By suitably modifying the corresponding result for $H^p(\mathbf{R}^n)$ spaces, we can establish the following lemma.

Lemma 2.3

Let $f \in h^p(\mathbf{R}^n)$, with $0 < p \leq \infty$ and let $\Phi \in \mathcal{S}(\mathbf{R}^n)$. Then $M_\Phi^* f \in L^p(\mathbf{R}^n)$ and there exists a positive constant C such that

$$\|M_\Phi^* f\|_{L^p(\mathbf{R}^n)} \leq C \|f\|_{L^p(\mathbf{R}^n)}.$$

Recall that a tempered distribution f is said to be bounded if $f * \Phi \in L^\infty(\mathbf{R}^n)$ whenever $\Phi \in \mathcal{S}(\mathbf{R}^n)$. The next lemma shows that any $f \in h^p(\mathbf{R}^n)$, with $0 < p \leq 1$, has this property.

Lemma 2.4

If $f \in h^p(\mathbf{R}^n)$, with $0 < p \leq 1$, then it is restricted at infinity.

Proof. Assume $f \in h^p(\mathbf{R}^n)$, with $0 < p \leq 1$ and let Φ be any Schwartz function, we prove that $f * \Phi$ belongs to $L^q(\mathbf{R}^n)$, for every $q \geq p$. Let $M_\Phi^* f$ be the nontangential maximal function of f defined in (7), since $|f * \Phi(x)| \leq M_\Phi^* f(x)$, it follows immediately that $f * \Phi \in L^p(\mathbf{R}^n)$. Moreover, by the definition of M_Φ^* , we have

$$|(f * \Phi)(x)| \leq M_\Phi^* f(y), \quad \text{for every } y \text{ with } |y - x| \leq 1.$$

Hence, taking the p^{th} power of both sides of the previous inequality and integrating over the ball $B(x, 1)$ with respect to y , by Lemma 2.3 we get

$$\begin{aligned} |(f * \Phi)(x)|^p &\leq \frac{1}{|B(x, 1)|} \int_{B(x, 1)} (M_\Phi^* f(y))^p dy \\ &\leq C \|M_\Phi^* f\|_{h^p}^p \leq C \|f\|_{h^p}^p, \end{aligned}$$

so $f * \Phi \in L^\infty(\mathbf{R}^n)$. Since $f * \Phi \in L^p(\mathbf{R}^n) \cap L^\infty(\mathbf{R}^n)$, we obtain that $f * \Phi \in L^q(\mathbf{R}^n)$. $\square$

We will need a result of approximation in $h^p(\mathbf{R}^n)$ whose proof requires the following lemma.

Lemma 2.5

Let $\Phi, \Psi \in \mathcal{S}(\mathbf{R}^n)$, and let Λ be a finite set of multi-indices of $\mathbf{N}^n$. Then the finite collection of seminorms $\mathcal{F} = \{\|\cdot\|_{\alpha, \beta} : \alpha, \beta \in \Lambda\}$ is bounded on the set $\{\Phi * \Psi_s : 0 < s \leq 1\}$.

Proof. Let $\Phi, \Psi \in \mathcal{S}(\mathbf{R}^n)$. By a change of variable, we can easily see that

$$|x^\alpha \partial^\beta (\Phi * \Psi_s)(x)| \leq |x|^{|\alpha|} \int_{\mathbf{R}^n} |(\partial^\beta \Phi)(x - sz)| |\Psi(z)| dz.$$

If $|x| \leq 1$, we have

$$|x|^{|\alpha|} \int_{\mathbf{R}^n} |(\partial^\beta \Phi)(x - sz)| |\Psi(z)| dz \leq \max_{\alpha, \beta \in \Lambda} \|\Phi\|_{\alpha, \beta} \|\Psi\|_{L^1(\mathbf{R}^n)}.$$

If $|x| \geq 1$ we write

$$\begin{aligned} |x|^{|\alpha|} \int_{\mathbf{R}^n} |(\partial^\beta \Phi)(x - sz)| |\Psi(z)| dz &= |x|^{|\alpha|} \int_{|z| \leq \frac{|x|}{2}} |(\partial^\beta \Phi)(x - sz)| |\Psi(z)| dz \\ &\quad + |x|^{|\alpha|} \int_{|z| > \frac{|x|}{2}} |(\partial^\beta \Phi)(x - sz)| |\Psi(z)| dz \\ &= I_1 + I_2. \end{aligned}$$

To estimate the integral I_1 we observe that since $|z| \leq \frac{|x|}{2}$ and $0 < s \leq 1$, then $|x - sz| \geq \frac{|x|}{2}$. Moreover, since $\Phi \in \mathcal{S}(\mathbf{R}^n)$, then $|(\partial^\beta \Phi)(x - sz)| \leq C|x|^{-M}$ for every $M \in \mathbf{N}$, for every $\beta \in \Lambda$ and for a positive constant C depending on M . Since $|x| > 1$, by choosing an integer $M \geq \max_{\alpha \in \Lambda} |\alpha|$ we get

$$I_1 \leq C|x|^{|\alpha| - M} \int_{|z| \leq \frac{|x|}{2}} |\Psi(z)| dz \leq C\|\Psi\|_{L^1(\mathbf{R}^n)}.$$

To estimate the integral I_2 we see that $|\Psi(z)| \leq C(1+|z|)^{-N}$ for every $N \in \mathbf{N}$ and for a positive constant C depending on N , because $\Psi \in \mathcal{S}(\mathbf{R}^n)$. Therefore, by choosing an integer $N \geq n + \max_{\alpha \in \Lambda} |\alpha|$ we get

$$I_2 \leq C \max_{\alpha, \beta \in \Lambda} \|\Phi\|_{\alpha, \beta} |x|^{|\alpha|} \int_{|x| > \frac{|x|}{2}} |z|^{-N} dz \leq C \max_{\alpha, \beta \in \Lambda} \|\Phi\|_{\alpha, \beta}.$$

So for $|x| > 1$ we have

$$|x|^{|\alpha|} \int_{\mathbf{R}^n} |(\partial^\beta \Phi)(x - sz)| |\Psi(z)| dz \leq C$$

for every $\alpha, \beta \in \Lambda$.

The previous estimates then show that $\|\Phi * \Psi_s\|_{\alpha, \beta} \leq C$, for every $\alpha, \beta \in \Lambda$, i.e. the finite collection of seminorms $\mathcal{F}$ is bounded on the set $\{\Phi * \Psi_s : 0 < s \leq 1\}$. $\square$

We can state the following proposition.

Proposition 2.6

Let $f \in h^p(\mathbf{R}^n)$ with $0 < p \leq 1$ and let $\Phi \in \mathcal{S}(\mathbf{R}^n)$ with $\int \Phi dx = 1$. Then the two following properties hold:

- (i) there exists a positive constant c such that $\|f * \Phi_t\|_{h^p(\mathbf{R}^n)} \leq c\|f\|_{h^p(\mathbf{R}^n)}$ uniformly in t for $0 < t \leq 1$;
- (ii) $f * \Phi_t \rightarrow f$ in h^p as $t \rightarrow 0^+$.

Proof. Let $f \in h^p(\mathbf{R}^n)$ with $0 < p \leq 1$, let $\Psi \in \mathcal{S}(\mathbf{R}^n)$ with $\int \Psi dx = 1$ and let $M_\Psi f$ be the maximal function defined in (3). We prove that there exists a positive constant C such that

$$\|f * \Phi_t\|_{h^p(\mathbf{R}^n)}^p = \|M_\Psi(f * \Phi_t)\|_{L^p(\mathbf{R}^n)}^p \leq C\|f\|_{h^p(\mathbf{R}^n)}^p$$

for every t with $0 < t \leq 1$. By a change of variable we can see that

$$(\Phi_t * \Psi_s)(x) = (\Phi * \Psi_{s/t})_t(x)$$

and, also

$$(\Phi_t * \Psi_s)(x) = (\Phi_{t/s} * \Psi)_s(x).$$

We write

$$(\Phi_t * \Psi_s)(x) = \begin{cases} (\Phi * \Psi_{s/t})_t(x) & \text{if } 0 < s < t \leq 1 \\ (\Phi_{t/s} * \Psi)_s(x) & \text{if } 0 < t \leq s \leq 1. \end{cases} \quad (8)$$

Now let Λ be a finite set of multi-indices of $\mathbf{N}^n$, $\mathcal{F} = \{\|\cdot\|_{\alpha, \beta} : \alpha, \beta \in \Lambda\}$ a finite collection of seminorms and $\mathcal{S}_{\mathcal{F}}$ the set of Schwartz functions defined in (4). Because of Lemma 2.5, we can find a positive constant c such that $c(\Phi * \Psi_{s/t}) \in \mathcal{S}_{\mathcal{F}}(\mathbf{R}^n)$ if $0 < s < t \leq 1$, and $c(\Phi_{t/s} * \Psi) \in \mathcal{S}_{\mathcal{F}}(\mathbf{R}^n)$ if $0 < t \leq s \leq 1$. Hence, if $0 < s < t \leq 1$ we can write

$$\begin{aligned} |(f * \Phi_t * \Psi_s)(x)| &= c^{-1} \left| \left(c(\Psi_{s/t} * \Phi)_t * f \right)(x) \right| \\ &\leq c^{-1} \sup \{ |(\eta_t * f)(x)| : \eta \in \mathcal{S}_{\mathcal{F}}(\mathbf{R}^n), 0 < t \leq 1 \} \\ &= c^{-1} \mathcal{M}_{\mathcal{F}} f(x), \end{aligned}$$

where $\mathcal{M}_{\mathcal{F}}f$ is the grand maximal function defined in (5). An analogous estimate can also be proved in the case $0 < t \leq s \leq 1$ by using the second of the two equalities in (8). Therefore

$$\begin{aligned} \|f * \Phi_t\|_{h^p(\mathbf{R}^n)}^p &= \int_{\mathbf{R}^n} \left(\sup_{0 < s \leq 1} |(f * \Phi_t * \Psi_s)(x)| \right)^p dx \\ &\leq c^{-p} \int_{\mathbf{R}^n} (\mathcal{M}_{\mathcal{F}}f(x))^p dx \\ &= c^{-p} \|f\|_{h^p(\mathbf{R}^n)}^p \end{aligned}$$

for every t with $0 < t \leq 1$. This proves the assertion (i).

To conclude the proof we shall show that

$$\lim_{t \rightarrow 0^+} \|f * \Phi_t - f\|_{h^p(\mathbf{R}^n)}^p = \lim_{t \rightarrow 0^+} \|M_{\Psi}(f * \Phi_t - f)\|_{L^p(\mathbf{R}^n)}^p = 0.$$

This result is a consequence of Lebesgue's dominated convergence theorem if we prove that there exists a non-negative real-valued function $g \in L^1(\mathbf{R}^n)$ such that for every $0 < t \leq 1$

$$\left(\sup_{0 < s \leq 1} |(f * \Phi_t * \Psi_s - f * \Psi_s)(x)| \right)^p \leq g(x) \quad \text{for a.e. } x \in \mathbf{R}^n \quad (9)$$

and

$$\lim_{t \rightarrow 0^+} \sup_{0 < s \leq 1} |(f * \Phi_t * \Psi_s - f * \Psi_s)(x)| = 0 \quad \text{for a.e. } x \in \mathbf{R}^n. \quad (10)$$

We prove (9). By what we have previously proved

$$\begin{aligned} \sup_{0 < s \leq 1} |(f * \Phi_t * \Psi_s - f * \Psi_s)(x)| &\leq \sup_{0 < s \leq 1} |(f * \Phi_t * \Psi_s)(x)| + \sup_{0 < s \leq 1} |(f * \Psi_s)(x)| \\ &\leq c^{-1} \mathcal{M}_{\mathcal{F}}f(x) + M_{\Psi}f(x) \end{aligned}$$

for every t with $0 < t \leq 1$ and for any finite collection $\mathcal{F}$ of seminorms. Therefore condition (9) holds with $g(x) = (c^{-1} \mathcal{M}_{\mathcal{F}}f(x) + M_{\Psi}f(x))^p$ that belongs to $L^p(\mathbf{R}^n)$ because $f \in h^p(\mathbf{R}^n)$.

Finally we prove (10). We write $(\Phi_t * \Psi_s)(x)$ as in (8) and we observe that if $\mathcal{F}$ is any finite collection of seminorms then by Lemma 2.5, if $0 < s < t \leq 1$, $(\Phi * \Psi_{s/t}) \in \mathcal{S}_{\mathcal{F}}(\mathbf{R}^n)$ up to a positive constant, and $\int (\Phi * \Psi_{s/t})(x) dx = 1$. The same properties remain true also for the functions $(\Phi_{t/s} * \Psi)$ for $0 < t \leq s \leq 1$.

Next we decompose

$$\begin{aligned} \sup_{0 < s \leq 1} |(f * \Phi_t * \Psi_s - f * \Psi_s)(x)| &\leq \sup_{0 < s < t \leq 1} |(f * (\Phi * \Psi_{s/t})_t - f * \Psi_s)(x)| \\ &\quad + \sup_{0 < t \leq s \leq 1} |(f * (\Phi_{t/s} * \Psi)_s - f * \Psi_s)(x)|. \end{aligned}$$

Then we consider a Schwartz function η such that $\int \eta(x) dx = 1$ and $\int x^\alpha \eta(x) dx = 0$ for every $\alpha \in \mathbf{N}^n$ with $0 < |\alpha| \leq 1 + [n(1/p - 1)]$ and we decompose f as $f = (f - f * \eta) + f * \eta = f_1 + f_2$. As a consequence of Lemma 2.2 we have that

$f_1 \in H^p(\mathbf{R}^n)$, and by Lemma 2.4 we have that $f_2 \in L^\infty(\mathbf{R}^n) \cap C^\infty(\mathbf{R}^n)$. Hence we can write

$$\begin{aligned} \sup_{0 < s \leq 1} |(f * \Phi_t * \Psi_s - f * \Psi_s)(x)| &\leq \sup_{0 < s < t \leq 1} |(f_1 * (\Phi * \Psi_{s/t})_t - f_1 * \Psi_s)(x)| \\ &+ \sup_{0 < s < t \leq 1} |(f_2 * (\Phi * \Psi_{s/t})_t - f_2 * \Psi_s)(x)| \\ &+ \sup_{0 < t \leq s \leq 1} |(f_1 * (\Phi_{t/s} * \Psi)_s - f_1 * \Psi_s)(x)| \\ &+ \sup_{0 < t \leq s \leq 1} |(f_2 * (\Phi_{t/s} * \Psi)_s - f_2 * \Psi_s)(x)|. \end{aligned}$$

Property (10) now follows by standard arguments. $\square$

Among the properties that make the local Hardy spaces more suited to a great number of applications than the classical Hardy spaces, there is the fact that the Schwartz space $\mathcal{S}(\mathbf{R}^n)$ is dense in $h^p(\mathbf{R}^n)$. This property allows to localize elements in $h^p(\mathbf{R}^n)$ and its proof has been outlined in [2]; here we give the details for the sake of completeness. We start by some preliminary facts.

The following lemma can be obtained by modifying the analogous result for the corresponding nontangential maximal functions of functions defined on the upper half-space stated in [9, Chapter II.2.5].

Lemma 2.7

Let $F(x, t)$ be a Borel measurable function defined on the strip S . Assume that $F^*(x) = \sup_{|y-x| < t \leq 1/2} |F(y, t)| \in L^1(\mathbf{R}^n)$ and for any $a > 0$ set

$$F_a^*(x) = \sup \left\{ |F(y, t)| : |y - x| < at, 0 < t \leq \frac{1}{2} \right\}.$$

Then if $a \geq 1$ there exists a positive constant $C_{a,n}$ depending only on a and on the dimension n such that

$$\int_{\mathbf{R}^n} F_a^*(x) dx \leq C_{a,n} \int_{\mathbf{R}^n} F^*(x) dx.$$

As a consequence, if for any $a > 0$ and $\Phi \in \mathcal{S}(\mathbf{R}^n)$ with $\int \Phi dx \neq 0$ we set

$$\tilde{M}_\Phi f(x) = \sup \{ |(f * \Phi_t)(y)| : |y - x| < at, 0 < t \leq 1 \},$$

we have the following corollary.

Corollary 2.8

Let $a \geq 1$, $0 < p \leq 1$, $\Phi \in \mathcal{S}(\mathbf{R}^n)$, with $\int \Phi dx \neq 0$. Let f be a tempered distribution and let $M_\Phi^* f$ be the nontangential maximal function of f defined in (7). Then $\tilde{M}_\Phi f \in L^p(\mathbf{R}^n)$ if and only if $M_\Phi^* f \in L^p(\mathbf{R}^n)$.

Theorem 2.9

The Schwartz space $\mathcal{S}(\mathbf{R}^n)$ is dense in $h^p(\mathbf{R}^n)$ for $0 < p < \infty$.

Proof. It is easy to prove the first part of the statement. Now suppose we are given a tempered distribution $f \in h^p(\mathbf{R}^n)$, then we prove that there exists a family $\{f_{(R)}\}$ of Schwartz functions such that $\lim_{R \rightarrow +\infty} \|f - f_{(R)}\|_{h^p(\mathbf{R}^n)} = 0$. Let $\Phi \in \mathcal{S}(\mathbf{R}^n)$ with $\int_{\mathbf{R}^n} \Phi dx = 1$, because of Proposition 2.6, given any $\varepsilon > 0$ we can find a value $t_0 \in (0, 1)$ such that $\|f * \Phi_{t_0} - f\|_{h^p(\mathbf{R}^n)} < \varepsilon$. Let $\eta \in \mathcal{S}(\mathbf{R}^n)$ be a function satisfying $\eta(x) = 1$ for $|x| \leq 1$ and $\eta(x) = 0$ for $|x| \geq 2$. For $R > 1$ we set

$$\eta_{(R)}(x) = \eta\left(\frac{x}{R}\right) \quad \text{and} \quad f_{(R)}(x) = \eta_{(R)}(x)(f * \Phi_{t_0})(x).$$

We notice that $f_{(R)} \in \mathcal{S}(\mathbf{R}^n)$ for all $R > 0$ and

$$\begin{aligned} \|f - f_{(R)}\|_{h^p(\mathbf{R}^n)}^p &= \|\eta_{(R)}(f * \Phi_{t_0}) - f\|_{h^p(\mathbf{R}^n)}^p \\ &\leq \|\eta_{(R)}(f * \Phi_{t_0}) - f * \Phi_{t_0}\|_{h^p(\mathbf{R}^n)}^p + \|f * \Phi_{t_0} - f\|_{h^p(\mathbf{R}^n)}^p \\ &< \|\eta_{(R)}(f * \Phi_{t_0}) - f * \Phi_{t_0}\|_{h^p(\mathbf{R}^n)}^p + \varepsilon^p. \end{aligned}$$

Therefore, to prove that $\lim_{R \rightarrow +\infty} \|f - f_{(R)}\|_{h^p(\mathbf{R}^n)} = 0$ it is sufficient to show that

$$\lim_{R \rightarrow +\infty} \|\eta_{(R)}(f * \Phi_{t_0}) - f * \Phi_{t_0}\|_{h^p(\mathbf{R}^n)} = 0. \quad (11)$$

Let $Q\left(0, \frac{R}{2\sqrt{n}}\right)$ be the cube centered at the origin and of side $\frac{R}{2\sqrt{n}}$ and let Ψ be a smooth compactly supported function whose support is contained in the ball $B(0, 1/2)$, with $\int \Psi dx = 1$. Then

$$\begin{aligned} &\|\eta_{(R)}(f * \Phi_{t_0}) - f * \Phi_{t_0}\|_{h^p(\mathbf{R}^n)}^p \\ &= \int_{\mathbf{R}^n} \left(\sup_{0 < s \leq 1} \left| \left(\Psi_s * (\eta_{(R)}(f * \Phi_{t_0}) - f * \Phi_{t_0}) \right) (x) \right| \right)^p dx \\ &= \int_{Q\left(0, \frac{R}{2\sqrt{n}}\right)} \left(\sup_{0 < s \leq 1} \left| \left(\Psi_s * (\eta_{(R)} - 1)(f * \Phi_{t_0}) \right) (x) \right| \right)^p dx \\ &\quad + \int_{^cQ\left(0, \frac{R}{2\sqrt{n}}\right)} \left(\sup_{0 < s \leq 1} \left| \left(\Psi_s * (\eta_{(R)} - 1)(f * \Phi_{t_0}) \right) (x) \right| \right)^p dx. \end{aligned}$$

We can easily check that the integral over $Q\left(0, \frac{R}{2\sqrt{n}}\right)$ is zero. To estimate the integral over $^cQ\left(0, \frac{R}{2\sqrt{n}}\right)$ we write $^cQ\left(0, \frac{R}{2\sqrt{n}}\right) = \cup_j Q_j$ where the Q_j 's are cube of unitary side whose interiors are disjoint, so that

$$\begin{aligned} &\int_{^cQ\left(0, \frac{R}{2\sqrt{n}}\right)} \left(\sup_{0 < s \leq 1} \left| \left(\Psi_s * (\eta_{(R)} - 1)(f * \Phi_{t_0}) \right) (x) \right| \right)^p dx \\ &= \sum_j \int_{Q_j} \left(\sup_{0 < s \leq 1} \left| \left(\Psi_s * (\eta_{(R)} - 1)(f * \Phi_{t_0}) \right) (x) \right| \right)^p dx. \end{aligned}$$

Since

$$\begin{aligned} \left| \left(\Psi_s * (\eta_{(R)} - 1)(f * \Phi_{t_0}) \right) (x) \right| &\leq \|\Psi_s\|_{L^1(\mathbf{R}^n)} \sup_{x \in Q_j} |(\eta_{(R)} - 1)(f * \Phi_{t_0})(x)| \\ &\leq \|\Psi\|_{L^1(\mathbf{R}^n)} (1 + \|\eta\|_{L^\infty(\mathbf{R}^n)}) \sup_{x \in Q_j} |(f * \Phi_{t_0})(x)| \\ &= C \sup_{x \in Q_j} |(f * \Phi_{t_0})(x)|, \end{aligned}$$

and $|Q_j| = 1$, we have

$$\begin{aligned} \int_{cQ\left(0, \frac{R}{2\sqrt{n}}\right)} \left(\sup_{0 < s \leq 1} \left| \left(\Psi_s * (\eta_{(R)} - 1)(f * \Phi_{t_0}) \right) (x) \right| \right)^p dx \\ \leq C^p \sum_j \sup_{x \in Q_j} |(f * \Phi_{t_0})(x)|^p \int_{Q_j} dx \\ = C^p \sum_j \sup_{x \in Q_j} |(f * \Phi_{t_0})(x)|^p. \end{aligned}$$

Now we see that

$$\sup_{x \in Q_j} |(f * \Phi_{t_0})(x)|^p \leq \sup_{\substack{|y-z| \leq \sqrt{n} \frac{s}{t_0} \\ 0 < s \leq 1}} |(f * \Phi_s)(y)|^p$$

for every $z \in Q_j$. Hence, by integrating both sides of the previous inequality over Q_j and taking into account the fact that $|Q_j| = 1$, we get

$$\sup_{x \in Q_j} |(f * \Phi_{t_0})(x)|^p \leq \int_{Q_j} \sup_{\substack{|y-z| \leq \sqrt{n} \frac{s}{t_0} \\ 0 < s \leq 1}} |(f * \Phi_s)(y)|^p dz.$$

Therefore we have proved that

$$\begin{aligned} \|\eta_{(R)}(f * \Phi_{t_0}) - f * \Phi_{t_0}\|_{h^p(\mathbf{R}^n)}^p &\leq C^p \sum_j \int_{Q_j} \sup_{\substack{|y-z| \leq \sqrt{n} \frac{s}{t_0} \\ 0 < s \leq 1}} |(f * \Phi_s)(y)|^p dz \\ &= C^p \int_{cQ\left(0, \frac{R}{2\sqrt{n}}\right)} \sup_{\substack{|y-z| \leq \sqrt{n} \frac{s}{t_0} \\ 0 < s \leq 1}} |(f * \Phi_s)(y)|^p dz \\ &= C^p \int_{cQ\left(0, \frac{R}{2\sqrt{n}}\right)} (\tilde{M}_\Phi f(z))^p dz. \end{aligned}$$

Since $f \in h^p(\mathbf{R}^n)$, by Lemma 2.3 $M_\Phi^* f \in L^p(\mathbf{R}^n)$ and so, by Corollary 2.8 $\tilde{M}_\Phi f \in L^p(\mathbf{R}^n)$. Therefore

$$\lim_{R \rightarrow +\infty} \int_{cQ\left(0, \frac{R}{2\sqrt{n}}\right)} (\tilde{M}_\Phi f(z))^p dz = 0$$

which implies (11) and concludes the proof. $\square$

Finally, we conclude this section by recalling an important boundedness result for pseudo-differential operators on $h^p(\mathbf{R}^n)$ whose proof can be found in [2].

Theorem 2.10

Pseudo-differential operators of order zero are bounded on $h^p(\mathbf{R}^n)$ for $0 < p \leq 1$.

3. Local Hardy spaces and generalized Cauchy-Riemann equations

Many of the results of the theory of local Hardy spaces $h^p(\mathbf{R}^n)$, including their characterization in terms of local Riesz transforms in the case $1 - 1/n < p \leq 1$, proved in [2], are based on the following construction. Suppose that we are given a vector $F(x, t) = (u_0(x, t), u_1(x, t), \dots, u_n(x, t))$ of $n + 1$ functions defined in some region of the Euclidean $(n + 1)$ -space satisfying the following generalized Cauchy-Riemann equations:

$$\sum_{j=0}^n \frac{\partial u_j}{\partial x_j}(x, t) = 0, \quad \text{and} \quad \frac{\partial u_j}{\partial x_k}(x, t) = \frac{\partial u_k}{\partial x_j}(x, t), \quad 0 \leq j, k \leq n. \quad (12)$$

Here, and at other times, it will be convenient to set $t = x_0$. Then, for $1 - \frac{1}{n} < p \leq 1$, $|F|^p$ is subharmonic.

In order to prove the characterization of $h^p(\mathbf{R}^n)$ spaces by the local Riesz transforms in the general case $0 < p \leq 1$, we need to extend the previous property to all $p > 0$. This can be done in the following way, see [10] or [9, Chapter III.5.16]

When $n > 1$ and $M \geq 1$ is an integer, we consider the tensor product of M copies of $\mathbf{R}^{n+1}$, that is $\bigotimes^M \mathbf{R}^{n+1}$. This is a vector space of dimension $(n+1)^M$ and its elements are called tensors of rank M . More precisely, if $e_0, \dots, e_n$ are the unit vectors of the standard basis of $\mathbf{R}^{n+1}$, then

$$\bigotimes^M \mathbf{R}^{n+1} = \left\{ F = \sum_I F_I e^{\otimes I} : I = (i_1, \dots, i_M) \text{ for } i_k \in \{0, \dots, n\}, k = 1, \dots, M \right\}$$

where $e^{\otimes I} = e_{i_1} \otimes \dots \otimes e_{i_M}$. The tensors of $\bigotimes^M \mathbf{R}^{n+1}$ inherit the usual inner product from $\mathbf{R}^{n+1}$. Hence, if

$$F, G \in \bigotimes^M \mathbf{R}^{n+1}, F = \sum_I F_I(x, t) e^{\otimes I}, G = \sum_J G_J(x, t) e^{\otimes J}$$

then

$$\langle F, G \rangle = \sum_I \sum_J F_I G_J \langle e^{\otimes I}, e^{\otimes J} \rangle = \sum_I F_I G_I.$$

We will consider tensor-valued functions

$$F : \mathbf{R}^{n+1} \rightarrow \bigotimes^M \mathbf{R}^{n+1}, F(x, t) = \sum_I F_I(x, t) e^{\otimes I}.$$

DEFINITION 3.1 Let $F : \mathbf{R}^{n+1} \rightarrow \bigotimes^M \mathbf{R}^{n+1}$, be the tensor-valued function $F(x, t) = \sum_I F_I(x, t) e^{\otimes I}$. We say that

- (i) F is symmetric if $F(x, t) = \sum_I F_I(x, t) e^{\otimes \sigma(I)}$ for every σ belonging to the group Σ_M of permutations on M elements;

(ii) F has trace zero if it is orthogonal to all elements of the form

$$\begin{aligned} & \sum_{j=0}^n \sum_{(j_1, \dots, j_{M-2})} G_{(j_1, \dots, j_{M-2})}(x, t) e_j \otimes e_j \otimes e_{j_1} \otimes \dots \otimes e_{j_{M-2}}, \\ & \sum_{j=0}^n \sum_{(j_1, \dots, j_{M-2})} G_{(j_1, \dots, j_{M-2})}(x, t) e_j \otimes e_{j_1} \otimes e_j \otimes \dots \otimes e_{j_{M-2}}, \\ & \dots \dots \dots \\ & \sum_{j=0}^n \sum_{(j_1, \dots, j_{M-2})} G_{(j_1, \dots, j_{M-2})}(x, t) e_{j_1} \otimes \dots \otimes e_{j_{M-2}} \otimes e_j \otimes e_j. \end{aligned}$$

It can be easily seen that F is symmetric if and only if $F_I(x, t) = F_{\sigma(I)}(x, t)$ for every $\sigma \in \Sigma_M$. If F is symmetric it can be proved that F has trace zero if and only if

$$\sum_{j=0}^n F_{(j, j, i_3, \dots, i_M)}(x, t) = 0$$

for all $i_3, \dots, i_M \in \{0, \dots, n\}$.

For $F : \mathbf{R}^{n+1} \rightarrow \bigotimes^M \mathbf{R}^{n+1}$,

$$F(x, t) = \sum_I F_I(x, t) e^{\otimes I},$$

we define its gradient $\nabla F : \mathbf{R}^{n+1} \rightarrow \bigotimes^{M+1} \mathbf{R}^{n+1}$ as

$$\nabla F(x, t) = \sum_{j=0}^n \frac{\partial F}{\partial x_j}(x, t) \otimes e_j = \sum_{(i_1, \dots, i_M, j)} \frac{\partial F_I}{\partial x_j}(x, t) e^{\otimes (i_1, \dots, i_M, j)}.$$

DEFINITION 3.2 Let $F : \mathbf{R}^{n+1} \rightarrow \bigotimes^M \mathbf{R}^{n+1}$ be the tensor-valued function

$$F(x, t) = \sum_I F_I(x, t) e^{\otimes I}.$$

We say that F satisfies the generalized Cauchy-Riemann equations if F is symmetric and has trace zero and ∇F is also symmetric, i.e.

$$\partial_{x_k} F_I(x, t) = \partial_{x_{i_l}} F_{(i_1, \dots, i_{l-1}, k, i_{l+1}, \dots, i_M)}(x, t) \quad \text{for all } l = 1, \dots, M, \text{ for all } k = 0, \dots, n$$

and has trace zero, i.e.

$$\sum_{j=0}^n \partial_{x_j} F_{(j, i_2, \dots, i_M)}(x, t) = 0 \quad \text{for all } i_2, \dots, i_M \in \{0, \dots, n\}.$$

When working with harmonic functions on the strip S it is often useful to use the notion of even and odd functions in S , that now we recall.

For a harmonic function u in the strip S , we say that

- (i) u is even if $u(x, t) = u(x, 1 - t)$ for every $(x, t) \in S$;
- (ii) u is odd if $u(x, t) = -u(x, 1 - t)$ for every $(x, t) \in S$.

We prove the following results.

Proposition 3.3

Let $n > 1$ and $M \geq 1$ be two integers, $\frac{n-1}{n-1+M} < p < \infty$ and $u(x, t)$ an even real-valued harmonic function in the strip S . Suppose that $F : S \rightarrow \bigotimes^M \mathbf{R}^{n+1}$ is a tensor-valued function of rank M defined by

$$F(x, t) = \sum_I F_I(x, t) e^{\otimes I}, \quad (x, t) \in S$$

where $I = (i_1, \dots, i_M)$ with $i_k \in \{0, \dots, n\}$ for $k = 1, \dots, M$, and satisfying the following properties:

- (i) $\langle F, \bigotimes^M e_0 \rangle = u$;
- (ii) F satisfies the generalize Cauchy-Riemann equations;
- (iii) there exists a positive constant A such that

$$\sup_{0 < t < 1} \int_{\mathbf{R}^n} |F(x, t)|^p dx \leq A;$$

- (iv) $F(x, t) = F(x, 1 - t)$ for every $(x, t) \in S$.

Then

$$u^*(x) = \sup_{|x-y| \leq t < \frac{1}{2}} |u(y, t)| \in L^p(\mathbf{R}^n)$$

and

$$\|u^*\|_{L^p(\mathbf{R}^n)} \leq C \sup_{0 < t < 1} \int_{\mathbf{R}^n} |F(x, t)|^p dx.$$

Proof. By (ii), the function $|F(x, t)|^p$ is subharmonic in the strip S for $\frac{n-1}{n-1+M} < p < \infty$, [10]. Because of its subharmonicity, the function $|F|^p$ satisfies the mean-value inequality at every point of S . This means that if we write $\min\{t, 1 - t\} = t^*$ and $\tilde{B} = \tilde{B}((x, t), t^*)$ is the $(n + 1)$ -dimensional ball contained in S , then

$$\begin{aligned} |F(x, t)|^p &\leq \frac{1}{|\tilde{B}|} \int_{\tilde{B}} |F(y, s)|^p dy ds \\ &\leq \frac{1}{|\tilde{B}|} \int_{t-t^*}^{t+t^*} \left(\int_{\mathbf{R}^n} |F(y, s)|^p dy \right) ds \\ &\leq \left(\sup_{0 < s < 1} \int_{\mathbf{R}^n} |F(y, s)|^p dy \right) \frac{1}{|\tilde{B}|} \int_{t-t^*}^{t+t^*} ds \\ &\leq A \frac{1}{|\tilde{B}|} \int_{t-t^*}^{t+t^*} ds \\ &= C(t^*)^{-n} \end{aligned}$$

from which we deduce that

$$|F(x, t)| \leq C(t^*)^{-n/p}.$$

Hence the function $|F(x, t)|$ is bounded in every closed strip

$$\overline{S}_\delta = \{(x, t) \in \mathbf{R}^{n+1} : \delta \leq t \leq 1 - \delta\} \quad \text{for } 0 < \delta < \frac{1}{2}.$$

For a fixed $\delta \in (0, \frac{1}{2})$, let $T_\delta : S \rightarrow S_\delta$ be the bijective mapping defined by

$$T_\delta(x, t) = (x, (1 - 2\delta)t + \delta), \quad (x, t) \in S$$

and let

$$G_\delta(x, t) = F(T_\delta(x, t)) = F(x, (1 - 2\delta)t + \delta), \quad (x, t) \in S.$$

The tensor G_δ satisfies the following properties:

- (a) $G_\delta(x, 0) = F(x, \delta)$ for every $x \in \mathbf{R}^n$;
- (b) $G_\delta(x, 1) = F(x, 1 - \delta)$ for every $x \in \mathbf{R}^n$;
- (c) $G_\delta(x, t) = G_\delta(x, 1 - t)$ for every $(x, t) \in S$;
- (d) $|G_\delta(x, t)|^p$ is a subharmonic function in S for $\frac{n-1}{n-1+M} < p < \infty$.

Since $|F(x, t)|^p$ is subharmonic in S and bounded in every closed strip $\overline{S}_\delta$, it follows that the function $|G_\delta(x, t)|^p$ is subharmonic and bounded in S and continuous in the closed strip $\overline{S} = \{(x, t) \in \mathbf{R}^{n+1} : 0 \leq t \leq 1\}$. The function $(|F(\cdot, \delta)|^p * P_t)(x)$, where P_t is the even Poisson kernel of S , is a harmonic function that is bounded on S and continuous on the closure of S . Moreover the boundary values of $|G_\delta(x, t)|^p$ and $(|F(\cdot, \delta)|^p * P_t)(x)$ agree, in fact, by using the properties of the tensor G_δ stated above and the fact that $\lim_{t \rightarrow 0^+} P_t = \lim_{t \rightarrow 1^-} P_t = \delta_0$, we have

$$\begin{aligned} \lim_{t \rightarrow 0^+} (|F(\cdot, \delta)|^p * P_t)(x) &= \lim_{t \rightarrow 1^-} (|F(\cdot, \delta)|^p * P_t)(x) = |F(x, \delta)|^p \\ &= |G_\delta(x, 0)|^p = |G_\delta(x, 1)|^p. \end{aligned}$$

Therefore, it follows that

$$|G_\delta(x, t)|^p \leq (|F(\cdot, \delta)|^p * P_t)(x) \quad (13)$$

holds for every $\delta \in (0, \frac{1}{2})$ and $(x, t) \in \overline{S}$.

Now assumption (iii) on F implies that the functions $|F(\cdot, \delta)|$ belong uniformly to $L^r(\mathbf{R}^n)$, where $r = \frac{p}{q} > 1$, provided we choose $\frac{n-1}{n-1+M} < q < p$. By the weak compactness of $L^r(\mathbf{R}^n)$, there exist a sequence $\{\delta_j\}$ with $\delta_j \rightarrow 0$ for $j \rightarrow +\infty$ and a function $h \in L^r(\mathbf{R}^n)$ such that $|F(\cdot, \delta_j)|^q \rightarrow h$ weakly in $L^r(\mathbf{R}^n)$, as $j \rightarrow +\infty$. By (13), we have

$$|G_{\delta_j}(x, t)|^q \leq (|F(\cdot, \delta_j)|^q * P_t)(x)$$

for $(x, t) \in \overline{S}$. The continuity of $|G_\delta(x, t)|^q$ on $\overline{S}$ implies that

$$\lim_{j \rightarrow +\infty} |G_{\delta_j}(x, t)|^q = \lim_{j \rightarrow +\infty} |F(x, (1 - 2\delta_j)t + \delta_j)|^q = |F(x, t)|^q. \quad (14)$$

Moreover, since $P_t(x) \in \mathcal{S}(\mathbf{R}^n)$ for $0 < t < 1$, we have that $|F(\cdot, \delta_j)|^q \rightarrow h$ weakly in $L^r(\mathbf{R}^n)$ as $j \rightarrow +\infty$ and since $\mathcal{S}(\mathbf{R}^n) \subset L^{r'}(\mathbf{R}^n)$ where $\frac{1}{r} + \frac{1}{r'} = 1$, we have that

$$\lim_{j \rightarrow +\infty} (|F(\cdot, \delta_j)|^q * P_t)(x) = (h * P_t)(x). \quad (15)$$

By (14) and (15) we get

$$|F(x, t)|^q \leq (h * P_t)(x).$$

We observe that $h \geq 0$ and that

$$\|h\|_{L^r(\mathbf{R}^n)}^r \leq \sup_{0 < t < 1} \int_{\mathbf{R}^n} |F(x, t)|^p dx$$

as a consequence of the fact that $|F(\cdot, \delta_j)|^q \rightarrow h$ weakly in $L^r(\mathbf{R}^n)$ as $j \rightarrow +\infty$, and of Hölder's inequality.

If we denote by $\mathcal{M}$ the standard Hardy-Littlewood maximal operator, we get

$$\begin{aligned} u^*(x) &= \sup_{|x-y| \leq t < \frac{1}{2}} |u(y, t)| = \sup_{|x-y| \leq t < \frac{1}{2}} |\langle F(y, t), \otimes^M e_0 \rangle| \\ &\leq \sup_{|x-y| \leq t < \frac{1}{2}} |F(y, t)| \leq \sup_{|x-y| \leq t < \frac{1}{2}} |(h * P_t)(y)|^{1/q} \leq C(\mathcal{M}h(x))^{1/q}, \end{aligned}$$

where the last inequality is a consequence of [2, Lemma 2, page 31]. Finally,

$$\begin{aligned} \|u^*\|_{L^p(\mathbf{R}^n)}^p &= \int_{\mathbf{R}^n} (u^*(x))^p dx \\ &\leq C \int_{\mathbf{R}^n} (\mathcal{M}h(x))^{p/q} dx \\ &\leq C \|h\|_{L^r(\mathbf{R}^n)}^r \\ &\leq \sup_{0 < t < 1} \int_{\mathbf{R}^n} |F(x, t)|^p dx. \end{aligned}$$

This proves the proposition. $\square$

Clearly, one can prove the odd version of the above proposition in a complete analogous manner.

Corollary 3.4

Let $n > 1$ and $M \geq 1$ be two integers, let $\frac{n-1}{n-1+M} < p < \infty$ and let $u(x, t)$ be a real-valued harmonic function in the strip S . Suppose that $F : S \rightarrow \otimes^M \mathbf{R}^{n+1}$ is a tensor-valued function of rank M defined by

$$F(x, t) = \sum_I F_I(x, t) e^{\otimes I}, \quad (x, t) \in S$$

where $I = (i_1, \dots, i_M)$ with $i_k \in \{0, \dots, n\}$ for $k = 1, \dots, M$, and satisfying the following properties

- (i) $\langle F, \otimes^M e_0 \rangle = u$;
- (ii) F satisfies the generalized Cauchy-Riemann equations;
- (iii) there exists a positive constant A such that

$$\sup_{0 < t < 1} \int_{\mathbf{R}^n} |F(x, t)|^p dx \leq A.$$

Then

$$u^*(x) := \sup \left\{ |u(y, t)| : |x - y| \leq t, \text{ if } 0 < t \leq \frac{1}{2}, |x - y| \leq 1 - t, \text{ if } \frac{1}{2} < t < 1 \right\} \in L^p(\mathbf{R}^n)$$

and

$$\|u^*\|_{L^p(\mathbf{R}^n)} \leq C \sup_{0 < t < 1} \int_{\mathbf{R}^n} |F(x, t)|^p dx.$$

Proof. It suffices to write

$$u(x, t) = \frac{1}{2} (u(x, t) + u(x, 1 - t)) + \frac{1}{2} (u(x, t) + u(x, 1 - t)) =: u_p(x, t) + u_d(x, t)$$

and

$$F(x, t) = \frac{1}{2} (F(x, t) + F(x, 1 - t)) + \frac{1}{2} (F(x, t) + F(x, 1 - t)) =: F_p(x, t) + F_d(x, t).$$

The conclusion then follows by applying Proposition 3.3 with $u = u_p$ and $F = F_p$ and its odd “version” with $u = u_d$ and $F = F_d$. $\square$

4. Characterization by the local Riesz transforms

In order to state our result concerning the characterization of the local Hardy spaces in terms of the local Riesz transforms, we need some preliminary facts.

For $k = 0, \dots, n$ let $\tilde{r}_k$ be the operators defined as follows:

$$(\tilde{r}_k f)^\wedge(\xi) = -i \frac{\xi_k}{|\xi|} \tanh(\pi|\xi|) \hat{f}(\xi), \quad k = 1, \dots, n \quad (16)$$

and

$$(\tilde{r}_0 f)^\wedge(\xi) = \tanh(\pi|\xi|) \hat{f}(\xi). \quad (17)$$

We denote by $\check{f}$ the inverse Fourier transform of a tempered distribution f and we state the following lemma.

Lemma 4.1

Let Ψ be a fixed Schwartz function with $\int \Psi dx = 1$, and $M > 0$ an integer. Suppose $(n - 1)/(n + M - 1) < p \leq 1$ and let f be a tempered distribution which is restricted at infinity. If there exists a constant $A > 0$ such that for all ε with $0 < \varepsilon \leq 1$ we have

$$\sum_{M \leq |\alpha| \leq M+1} \|\tilde{r}^\alpha(f) * \Psi_\varepsilon\|_{L^p(\mathbf{R}^n)} \leq A, \quad (18)$$

then $\tilde{r}_0^M(f) \in h^p(\mathbf{R}^n)$, where $\alpha = (\alpha_0, \dots, \alpha_n)$ is a $n + 1$ -tuple of natural numbers.

Proof. We shall use the following notation. If l is a fixed positive integer and $J = (j_1, \dots, j_l)$ with $j_k \in \{0, \dots, n\}$ for $k = 1, \dots, l$, we denote by $\alpha(J) = (\alpha_{0,J}, \dots, \alpha_{n,J})$ the vector of $\mathbf{R}^{n+1}$ whose component $\alpha_{m,J}$ is given by the number of the components of J that are equal to m , for $m = 0, \dots, n$. We also denote $\ell(\alpha(J)) = \alpha_{1,J} + \dots + \alpha_{n,J}$.

Let P_t be the even Poisson kernel of the strip S defined in (6) and let Q_t be the kernel on S whose Fourier transform is

$$\widehat{Q}_t(\xi) = \frac{\sinh((1-2t)\pi|\xi|)}{\cosh(\pi|\xi|)}.$$

We observe that $\widehat{Q}_t(\xi)$ is rapidly decreasing and fails to be smooth at the origin where it is only continuous; in addition, $\left(\partial_t^2 + \sum_{j=1}^n \partial_{x_j}^2\right) Q_t(x) = 0$ as we can easily see by taking the Fourier transforms in the x -variable of both sides.

Moreover, it is easy to see that

$$\widehat{\partial_t(Q_t)}(\xi) = -2\pi|\xi|\widehat{P}_t(\xi) \quad \text{and} \quad \widehat{\partial_t(P_t)}(\xi) = -2\pi|\xi|\widehat{Q}_t(\xi). \quad (19)$$

Setting $f_\varepsilon = f * \Psi_\varepsilon$, and

$$K_t(x) = \begin{cases} P_t(x) & \text{if } \ell(\alpha(I)) \text{ is even} \\ Q_t(x) & \text{if } \ell(\alpha(I)) \text{ is odd,} \end{cases}$$

we define a tensor-valued function $F_\varepsilon : S \rightarrow \bigotimes^M \mathbf{R}^{n+1}$, $F_\varepsilon(x, t) = \sum_I F_{I,\varepsilon}(x, t) e^{\otimes I}$ where

$$F_{I,\varepsilon}(x, t) = (\tilde{r}^{\alpha(I)}(f_\varepsilon) * K_t)(x)$$

and $I = (i_1, \dots, i_M)$ with $i_k \in \{0, \dots, n\}$ for $k = 1, \dots, M$.

We show that the tensor F_ε is symmetric and has trace zero.

Since $\alpha(I) = \alpha(\sigma(I))$ for every σ belonging to the group of permutations on M letters, we have that $F_{I,\varepsilon}(x, t) = F_{\sigma(I),\varepsilon}(x, t)$, hence F_ε is symmetric.

To verify that F_ε has trace zero it suffices to prove that

$$\sum_{j=0}^n F_{(j,j,i_3,\dots,i_M),\varepsilon}(x, t) = 0$$

for all the indexes $i_3, \dots, i_M \in \{0, \dots, n\}$. Since

$$\alpha(j, j, i_3, \dots, i_M) = \alpha(j, j) + \alpha(i_3, \dots, i_M) = 2e_j + \alpha(i_3, \dots, i_M),$$

where e_j is the j^{th} unit vector of the standard basis of $\mathbf{R}^{n+1}$, and for every $j = 1, \dots, n$, $\ell(\alpha(0, 0, i_3, \dots, i_M))$ is even if and only if $\ell(\alpha(j, j, i_3, \dots, i_M))$ is even, by taking the Fourier transform in the x -variable we get

$$\begin{aligned} \left(\sum_{j=0}^n F_{(j,j,i_3,\dots,i_M),\varepsilon}(\cdot, t) \right)^\wedge(\xi) &= \left(\sum_{j=0}^n \tilde{r}_j^2 \tilde{r}^{\alpha(i_3,\dots,i_M)}(f_\varepsilon) * K_t \right)^\wedge(\xi) \\ &= \left(\tanh^2(\pi|\xi|) + \sum_{j=1}^n \left(-i \frac{\xi_j}{|\xi|} \tanh(\pi|\xi|) \right)^2 \right) \\ &\quad \times \left(\tilde{r}^{\alpha(i_3,\dots,i_M)}(f_\varepsilon) * K_t \right)^\wedge(\xi) = 0 \end{aligned}$$

from which we deduce that F_ε has trace zero.

Now, let $\nabla F_\varepsilon : S \rightarrow \bigotimes^{M+1} \mathbf{R}^{n+1}$ be the gradient of F_ε

$$\nabla F_\varepsilon(x, t) = \sum_{(I, j)} \frac{\partial F_{I, \varepsilon}}{\partial x_j}(x, t) e^{\otimes(I, j)}.$$

As we have already said, it will be convenient to set $x_0 = t$.

We prove that ∇F_ε is symmetric and has trace zero.

Set $I' = (i_1, \dots, i_{k-1}, j, i_{k+1}, \dots, i_M)$, to show that ∇F_ε is symmetric we need to verify that $\partial_{x_j} F_{I, \varepsilon}(x, t) = \partial_{x_{i_k}} F_{I', \varepsilon}(x, t)$ for every $k = 1, \dots, M$ and for every $j = 0, \dots, n$. If $j = i_k$ there is nothing to prove. Hence, we first assume $j \neq i_k$ and $j, i_k \geq 1$. In this case, since $\ell(\alpha(I'))$ is even if and only if $\ell(\alpha(I))$ is even, and since $\alpha(I') = \alpha(I) + e_j - e_{i_k}$, by taking the Fourier transform in the x -variable, we get

$$\begin{aligned} \left(\partial_{x_j} F_{I, \varepsilon}(\cdot, t) \right)^\wedge(\xi) &= 2\pi i \xi_j \left(\tilde{r}^{\alpha(I)}(f_\varepsilon) * K_t \right)^\wedge(\xi) \\ &= \frac{2\pi \xi_j \xi_{i_k}}{|\xi|} \tanh(\pi|\xi|) \left(\tilde{r}^{\alpha(I) - e_{i_k}}(f_\varepsilon) * K_t \right)^\wedge(\xi) \end{aligned}$$

and

$$\begin{aligned} \left(\partial_{x_{i_k}} F_{I', \varepsilon}(\cdot, t) \right)^\wedge(\xi) &= 2\pi i \xi_{i_k} \left(\tilde{r}^{\alpha(I) + e_j - e_{i_k}}(f_\varepsilon) * K_t \right)^\wedge(\xi) \\ &= \frac{2\pi \xi_j \xi_{i_k}}{|\xi|} \tanh(\pi|\xi|) \left(\tilde{r}^{\alpha(I) - e_{i_k}}(f_\varepsilon) * K_t \right)^\wedge(\xi). \end{aligned}$$

Then, we assume that $j = 0$ and $i_k \geq 1$ (the case $j \geq 1$ and $i_k = 0$ is similar). In this case $\alpha(I') = (\alpha_{0, I} + 1, \alpha_{1, I}, \dots, \alpha_{n, I}) - e_{i_k}$, but if $\ell(\alpha(I))$ is even, then $\ell(\alpha(I'))$ is odd and viceversa. Let us consider the case in which $\ell(\alpha(I))$ is even (the case in which $\ell(\alpha(I))$ is odd can be treated analogously). In this case, by taking the Fourier transform in the x -variable and recalling the formulas in (19) we obtain

$$\begin{aligned} \left(\partial_{x_j} F_{I, \varepsilon}(\cdot, t) \right)^\wedge(\xi) &= \left(\tilde{r}^{\alpha(I)}(f_\varepsilon) \right)^\wedge(\xi) (\partial_t P_t)^\wedge(\xi) \\ &= \left(-i \frac{\xi_{i_k}}{|\xi|} \tanh(\pi|\xi|) \right) \left(\tilde{r}^{\alpha(I) - e_{i_k}}(f_\varepsilon) \right)^\wedge(\xi) \left(-2\pi|\xi| \widehat{Q}_t(\xi) \right) \\ &= 2\pi i \xi_{i_k} \tanh(\pi|\xi|) \left(\tilde{r}^{\alpha(I) - e_{i_k}}(f_\varepsilon) \right)^\wedge(\xi) \widehat{Q}_t(\xi) \end{aligned}$$

and

$$\begin{aligned} \left(\partial_{x_{i_k}} F_{I', \varepsilon}(\cdot, t) \right)^\wedge(\xi) &= 2\pi i \xi_{i_k} \left(\tilde{r}^{\alpha(I')}(f_\varepsilon) * Q_t \right)^\wedge(\xi) \\ &= 2\pi i \xi_{i_k} \left(\tilde{r}^{(\alpha_{0, I} + 1, \alpha_{1, I}, \dots, \alpha_{n, I}) - e_{i_k}}(f_\varepsilon) * Q_t \right)^\wedge(\xi) \\ &= 2\pi i \xi_{i_k} \tanh(\pi|\xi|) \left(\tilde{r}^{\alpha(I) - e_{i_k}}(f_\varepsilon) \right)^\wedge(\xi) \widehat{Q}_t(\xi) \end{aligned}$$

Hence, we can conclude that ∇F_ε is symmetric.

Finally, to prove that ∇F_ε has trace zero, we need to verify that

$$\sum_{j=0}^n \partial_{x_j} F_{(j, i_2, \dots, i_M), \varepsilon}(x, t) = 0$$

for every vector $(i_2, \dots, i_M)$ with $i_k \in \{0, \dots, n\}$, for $k = 2, \dots, M$.

Since if $\ell(\alpha((0, i_2, \dots, i_M)))$ is even then $\ell(\alpha((j, i_2, \dots, i_M)))$ is odd for $j = 1, \dots, n$, and viceversa, we shall distinguish between $\ell(\alpha((0, i_2, \dots, i_M)))$ even and odd. Suppose $\ell(\alpha((0, i_2, \dots, i_M)))$ is even (the case $\ell(\alpha((0, i_2, \dots, i_M)))$ odd being analogous). In this case, by taking the Fourier transform in the x -variable and recalling the formulas in (19) we get

$$\begin{aligned} & \left(\sum_{j=0}^n \partial_{x_j} F_{(j, i_2, \dots, i_M), \varepsilon}(\cdot, t) \right)^\wedge(\xi) \\ &= \left(\tilde{r}_0 \tilde{r}^{\alpha(i_2, \dots, i_M)}(f_\varepsilon) * \partial_t P_t \right)^\wedge(\xi) + \sum_{j=1}^n \partial_{x_j} \left(\tilde{r}_j \tilde{r}^{\alpha(i_2, \dots, i_M)}(f_\varepsilon) * Q_t \right)^\wedge(\xi) \\ &= \left(\tanh(\pi|\xi|) \widehat{\partial_t P_t}(\xi) + 2\pi|\xi| \widehat{Q_t}(\xi) \right) \left(\tilde{r}^{\alpha(i_2, \dots, i_M)}(f_\varepsilon) \right)^\wedge(\xi) = 0, \end{aligned}$$

which shows that ∇F_ε has trace zero. Thus we have proved that the tensor F_ε satisfies the generalized Cauchy-Riemann equations.

Since f is a tempered distribution which is restricted at infinity, we can prove that $|F_\varepsilon(x, t)|^p$ and $(|F_\varepsilon(\cdot, 0)|^p * P_t)(x)$ are bounded in S and continuous on $\bar{S}$. In addition, $|F_\varepsilon(x, t)|^p$ is subharmonic and $(|F_\varepsilon(\cdot, 0)|^p * P_t)(x)$ is harmonic on S and their boundary values agree. Hence it follows that

$$|F_\varepsilon(x, t)|^p \leq (|F_\varepsilon(\cdot, 0)|^p * P_t)(x) \quad (20)$$

hold for every $(x, t) \in \bar{S}$. By integrating both sides of the previous inequality with respect to x and observing that

$$F_{I, \varepsilon}(x, 0) = (\tilde{r}^{\alpha(I)}(f_\varepsilon) * K_0)(x) = \begin{cases} \tilde{r}_0^{\alpha_0} \circ \tilde{r}_1^{\alpha_1} \circ \dots \circ \tilde{r}_n^{\alpha_n}(f_\varepsilon) & \text{if } \ell(\alpha(I)) \text{ is even} \\ \tilde{r}_0^{\alpha_0+1} \circ \tilde{r}_1^{\alpha_1} \circ \dots \circ \tilde{r}_n^{\alpha_n}(f_\varepsilon) & \text{if } \ell(\alpha(I)) \text{ is odd,} \end{cases}$$

using (18) and the fact that $\|P_t\|_{L^1(\mathbf{R}^n)} = 1$ we obtain that

$$\int_{\mathbf{R}^n} |F_\varepsilon(x, t)|^p dx \leq \| |F_\varepsilon(\cdot, 0)|^p \|_{L^1(\mathbf{R}^n)} \|P_t\|_{L^1(\mathbf{R}^n)} \leq A.$$

Since $F_\varepsilon(x, t) \rightarrow F(x, t) = \sum_I F_I(x, t) e^{\otimes I}$, where $F_I(x, t) = (\tilde{r}^{\alpha(I)}(f) * K_t)(x)$, for each x and $0 < t \leq 1$, as $\varepsilon \rightarrow 0^+$, we can apply Fatou's lemma and conclude that

$$\sup_{0 < t < 1} \int_{\mathbf{R}^n} |F(x, t)|^p dx \leq A.$$

An application of Corollary 3.4 allows us to conclude that

$$\begin{aligned} u^*(x) &= \sup \left\{ |\langle F(y, t), \otimes^M e_0 \rangle| : |x - y| \leq t, \text{ if } 0 < t \leq \frac{1}{2}, |x - y| \leq 1 - t, \text{ if } \frac{1}{2} < t < 1 \right\} \\ &= \sup_{|x-y| \leq t < 1/2} |(\tilde{r}_0^M(f) * P_t)(y)| \in L^p(\mathbf{R}^n), \end{aligned}$$

hence, by Theorem 2.1 we have $\tilde{r}_0^M(f) \in h^p(\mathbf{R}^n)$. $\square$

For $k = 1, \dots, n$, let r_k be the local Riesz transform defined in (1) and let r_0 be the localized identity operator defined in (2). We are now in a position to establish our main result.

Theorem 4.2

Let Ψ be a fixed Schwartz function with $\int \Psi dx = 1$, let $M > 0$ and $n > 1$ be two integers and suppose $(n-1)/(n+M-1) < p \leq 1$. Then a tempered distribution f which is restricted at infinity belongs to $h^p(\mathbf{R}^n)$ if and only if

$$\theta * f \in h^p(\mathbf{R}^n) \quad (21)$$

and there exists a constant $A > 0$ such that for all ε with $0 < \varepsilon \leq 1$ we have

$$\sum_{M \leq |\alpha| \leq M+1} \|r^\alpha(f) * \Psi_\varepsilon\|_{L^p(\mathbf{R}^n)} \leq A. \quad (22)$$

Here $\alpha = (\alpha_0, \dots, \alpha_n)$ is a $n+1$ -tuple of natural numbers.

Proof. Suppose first that (21) and (22) hold. Setting

$$\left(\frac{1 - \hat{\theta}(\cdot)}{\tanh(\pi|\cdot|)} \right)^\sim(x) = \eta(x),$$

we can write

$$(r_k f)^\wedge(\xi) = (\tilde{r}_k(\eta * f))^\wedge(\xi), \quad k = 1, \dots, n$$

and also

$$(r_0 f)^\wedge(\xi) = (\tilde{r}_0(\eta * f))^\wedge(\xi)$$

where $\tilde{r}_k$ for $k = 0, \dots, n$ are the operators defined in (16) and (17). Therefore, by the assumption (22), if we set

$$g := (\hat{\eta}^M)^\sim * f,$$

we have that for all $\alpha \in \mathbf{N}^{n+1}$ with $M \leq |\alpha| \leq M+1$,

$$\tilde{r}^\alpha(g) * \Psi_\varepsilon = r^\alpha(f) * \Psi_\varepsilon \in L^p(\mathbf{R}^n)$$

uniformly in ε for $0 < \varepsilon \leq 1$. Hence, by Lemma 4.1, we conclude that

$$\left(\tanh^M(\pi|\cdot|) \hat{g}(\cdot) \right)^\sim \in h^p(\mathbf{R}^n)$$

from which we deduce that

$$\left((1 - \hat{\theta}(\cdot))^M \hat{f}(\cdot) \right)^\sim \in h^p(\mathbf{R}^n). \quad (23)$$

Since $f \mapsto f * \theta$ is a pseudo-differential operator of order zero, Theorem 2.10 implies that

$$(\hat{\theta}^m(\cdot) \hat{f}(\cdot))^\sim \in h^p(\mathbf{R}^n) \quad (24)$$

for every positive integer m . Combining (23) and (24) we obtain that $f \in h^p(\mathbf{R}^n)$.

To prove the converse, recall that if $f \in h^p(\mathbf{R}^n)$, then it is restricted at infinity by Lemma 2.4. In addition, an application of Theorem 2.10, shows that for all multi-indexes $\alpha \in \mathbf{N}^{n+1}$ with $M \leq |\alpha| \leq M+1$, $r^\alpha(f)$ and $\theta * f$ belong to $h^p(\mathbf{R}^n)$ since r_k , for $k = 0, \dots, n$, and $f \mapsto \theta * f$ are all pseudo-differential operators of order zero. Now, $|(r^\alpha(f) * \Psi_\varepsilon)(x)| \leq M_\Psi(r^\alpha f)(x)$, hence Theorem 2.1 shows that (22) holds, completing the proof of theorem. $\square$

By comparing this result with the classical characterization of the Hardy H^p spaces in terms of Riesz transforms, we notice that in the previous theorem we make the extra assumption $\theta * f \in h^p(\mathbf{R}^n)$. We remark that this assumption is necessary in order to control the behaviour of the distribution f at infinity. For, let $f = \hat{\mu}$ to be the Fourier transform of the distribution μ having support contained in a small ball centered at the origin, and such that $|f| \approx |x|^{-\delta}$ as $|x| \rightarrow +\infty$, for some small $\delta > 0$. Then, clearly f is C^∞ , $r^\alpha(f) = 0$ for all α , but $f \notin h^p(\mathbf{R}^n)$, since $M_\Phi f \notin L^p(\mathbf{R}^n)$.

We remark that if $n = 1$, Theorem 4.2 holds true without the assumption (21) and it has been proved in [2], by suitably modifying the arguments of [8, page 221] (see also [9, Chapter III.4.3]) for the analogous characterization of the classical Hardy spaces $H^p(\mathbf{R})$. This is the reason for us to restrict to the case $n > 1$.

References

1. C. Fefferman and E.M. Stein, H^p spaces of several variables, *Acta Math.* **129** (1972), 137–193.
2. D. Goldberg, A local version of real Hardy spaces, *Duke Math. J.* **46** (1979), 27–42.
3. A. Miyachi, On some Fourier multipliers for $H^p(\mathbf{R}^n)$, *J. Fac. Sci. Univ. Tokyo Sect. IA Math.* **27** (1980), 157–179.
4. A. Miyachi, On some singular Fourier multipliers, *J. Fac. Sci. Univ. Tokyo Sect. IA Math.* **28** (1981), 267–315.
5. A. Miyachi, Notes on Fourier multipliers for H^p , BMO and the Lipschitz spaces, *J. Fac. Sci. Univ. Tokyo Sect. IA Math.* **30** (1983), 221–242.
6. A. Miyachi, Products of distributions in H^p spaces, *Tohoku Math. J. (2)* **35** (1983), 483–498.
7. M.M. Peloso and S. Secco, Boundedness of Fourier integral operators on Hardy spaces, to appear on *Proc. of Edin. Math. Soc.*
8. E.M. Stein, *Singular Integrals and Differentiability Properties of Functions*, Princeton Mathematical Series **30**, Princeton University Press, Princeton, N.J., 1970.
9. E.M. Stein, *Harmonic Analysis: Real-Variable Methods, Orthogonality and Oscillatory Integrals*, Princeton Mathematical Series **43**, Princeton University Press, Princeton, N.J., 1993.
10. E.M. Stein and G. Weiss Generalization of the Cauchy-Riemann equations and representations of the rotation group, *Amer. J. Math.* **90** (1968), 163–196.