Warfield invariants in Abelian group algebras

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ABSTRACT

Let F be a field of characteristic $p \neq 0$ and G an Abelian group. For each prime q and each ordinal α we calculate in terms of G and its sections the Warfield q -invariants $W_{\alpha,q}(V(FG))$ of the group $V(FG)$ of all normalized units in the commutative group algebra FG when either $q = p$, G_t/G_p is infinite and F is perfect, or G is p -mixed. Surprisingly, these invariants do not depend on F .

This supplies recent result of ours in [3].

I. Introduction

Throughout the current article, suppose FG is the group algebra of an Abelian group G written multiplicatively over a field F of characteristic $p \neq 0$, and $V(FG)$ is its group consisting of all normalized units (often called normed units) that are invertible elements in FG with augmentation 1. For such a group G , the letter G_t will always denote the maximal torsion subgroup of G with q -component of torsion G_q one for every prime number q ; it is well-known that $G_t = \coprod_q G_q$ and if $G_t = G_p$ the group G is called p -mixed. For each ordinal α , we define the q^α -th power series of G as follows: $G^{q^0} = G$, $G^{q^\alpha} = (G^{q^{\alpha-1}})^q$ when α is non-limit and $G^{q^\alpha} = \bigcap_{\beta < \alpha} G^{q^\beta}$ when α is limit. For such a field F , the letter F^* designates the multiplicative group of F . As usual, for any set S , the symbol $|S|$ denotes the cardinality of S . All other notion and notation are, in general, standard and are in agreement with [4, 3], respectively.

In [3] we computed the Warfield p -invariants of $V(RG)$ when R is a perfect commutative ring with identity of characteristic $p \neq 0$ which has no zero divisors (we regretfully note that there are some bugs which are corrected below; for instance, in [3, Lemma 4] R has to be of nonzero characteristic as well as in [3, Corollary 7] R must be perfect).

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For completeness of the exposition and for readers' convenience, we recollect once again that for any ordinal number α the Warfield q -invariant of G is the cardinal

$$W_{\alpha,q}(G) = \text{rank}(G^{q^\alpha} / (G^{q^{\alpha+1}} G_q^{q^\alpha})).$$

It is clear that $W_{\alpha,q}(G) = W_{0,q}(G^{q^\alpha})$ and $W_{n,q}(G) = W_{n,q}(G/G_q)$ for all $0 \leq n < \omega$. This ordinal-to-cardinal function will be on the focus of our investigation in the present paper.

II. Preliminaries

Before stating and proving our main assertions we need some preliminary technicalities.

Lemma 1

$$W_{\alpha,q}(\coprod_{i \in I} G_i) = \sum_{i \in I} W_{\alpha,q}(G_i).$$

Proof. Observe that $(\coprod_{i \in I} G_i)^{q^\alpha} = \coprod_{i \in I} G_i^{q^\alpha}$, that $(\coprod_{i \in I} G_i)^{q^{\alpha+1}} = \coprod_{i \in I} G_i^{q^{\alpha+1}}$ and that $(\coprod_{i \in I} G_i)_q^{q^\alpha} = \coprod_{i \in I} (G_i)_q^{q^\alpha}$. Moreover,

$$\left(\coprod_{i \in I} G_i \right)^{q^{\alpha+1}} \left(\coprod_{i \in I} G_i \right)_q^{q^\alpha} = \prod_{i \in I} (G_i^{q^{\alpha+1}} (G_i)_q^{q^\alpha}).$$

Thus

$$\begin{aligned} \left(\coprod_{i \in I} G_i \right)^{q^\alpha} / \left[\left(\coprod_{i \in I} G_i \right)^{q^{\alpha+1}} \left(\coprod_{i \in I} G_i \right)_q^{q^\alpha} \right] &= \prod_{i \in I} G_i^{q^\alpha} / \prod_{i \in I} (G_i^{q^{\alpha+1}} (G_i)_q^{q^\alpha}) \\ &\cong \prod_{i \in I} (G_i^{q^\alpha} / (G_i^{q^{\alpha+1}} (G_i)_q^{q^\alpha})), \end{aligned}$$

and we wish only apply ([4, v. I, p. 105, Exercise 4]) to get the claim. $\square$

Recall that a commutative ring R with identity of prime characteristic p is said to be perfect if $R = R^p$, where $R^p = \{r^p | r \in R\}$ is the p -th power subring of R .

Lemma 2

Let $R \leq L$ be perfect commutative rings with the same identity of prime characteristic p . Then, for each ordinal $\alpha \geq 0$,

$$W_{\alpha,p}(V(RG)) \leq W_{\alpha,p}(V(LG))$$

Proof. Because of ([3, Lemma 1]) and the aforementioned property $W_{\alpha,p}(G) = W_{0,p}(G^{p^\alpha})$, we may consider only the case when $\alpha = 0$. We elementarily observe that

$$V(RG)V(LG^p)V_p(LG)/(V(LG^p)V_p(LG)) \subseteq V(LG)/(V(LG^p)V_p(LG)).$$

But we have that

$$\begin{aligned} V(RG)V(LG^p)V_p(LG)/(V(LG^p)V_p(LG)) &\cong V(RG)/(V(RG) \cap [V(LG^p)V_p(LG)]) \\ &= V(RG)/(V(RG^p)V_p(RG)). \end{aligned}$$

Indeed, if we choose $x \in V(RG) \cap [V(LG^p)V_p(LG)]$, then there exists a natural k such that

$$x^{p^k} \in V(RG^{p^k}) \cap V(LG^{p^{k+1}}) = V(RG^{p^{k+1}}) = V^{p^{k+1}}(RG).$$

Hence, it is readily seen that, $x \in V^p(RG)V_p(RG) = V(RG^p)V_p(RG)$ as required. $\square$

Remark. The same claim remains true even when R is isotype in L , that is, $R \cap L^{p^\alpha} = R^{p^\alpha}$ for each ordinal α , where R^{p^α} is defined as follows: $R^{p^0} = R$, $R^{p^\alpha} = (R^{p^{\alpha-1}})^p$ if $\alpha - 1$ exists and $R^{p^\alpha} = \bigcap_{\beta < \alpha} R^{p^\beta}$ otherwise. Nevertheless, for our further application, perfectness of rings is enough.

Lemma 3

Let $G_q = 1$. Then $G^{q^\omega} = G^{q^{\omega+1}}$.

Proof. Given $x \in G^{q^\omega}$, whence $x = g^q = a^{q^{n+1}}$ for every natural n . Therefore, $(ga^{-q^n})^q = 1$ and hence $g = a^{q^n} \in G^{q^n}$. That is why, $x \in (G^{q^\omega})^q = G^{q^{\omega+1}}$. $\square$

The next affirmation is pivotal.

Proposition 4 ([1])

If K is an algebraically closed field of $\text{char}(K) = p \neq 0$ and $G_p = 1$, then

$$V(KG) \cong \left(\prod_{|G_t|} G/G_t \right) \times \left(\prod_{\gamma} K^* \right),$$

where $\gamma = |G_t|$ if $|G_t|$ is infinite or $\gamma = |G_t| - 1$ if $|G_t|$ is finite.

As an immediate consequence of Lemma 1 and Proposition 4 we derive the following.

Proposition 5

If K is an algebraically closed field of $\text{char}(K) = p \neq 0$ and $G_p = 1$, then for each ordinal α and prime q

$$W_{\alpha,q}(V(KG)) = |G_t| \cdot W_{\alpha,q}(G/G_t).$$

Proof. Observe that $W_{\alpha,q}(K^*) = 0$ since K^* is divisible. Hereafter, we subsequently apply Proposition 4 along with Lemma 1 to obtain the wanted claim. $\square$

Remark. Notice that $W_{\alpha,q}(G/G_t) = W_{\alpha,q}(G)$ for any α such that $0 \leq \alpha < \omega$ and any prime q . Consequently, under the circumstances of Proposition 5, $W_{n,q}(V(KG)) = |G_t| \cdot W_{n,q}(G)$ for $n \in \mathbb{N} \cup \{0\}$. We ask the following:

Problem 1. Whether or not the inequalities $W_{\alpha,q}(G) \geq W_{\alpha,q}(G/G_q)$ and $W_{\alpha,q}(G) \geq W_{\alpha,q}(G/G_t)$ hold in general?

III. Main Results

Now we have at our disposal all the information necessary to proceed with the following statement.

Theorem 6

Suppose that F is a field of characteristic $p \neq 0$ and G is an Abelian group. Then, for every ordinal number α ,

- (1) $W_{\alpha,p}(V(FG)) = |G_t/G_p| \cdot W_{\alpha,p}(G)$,
provided F is perfect and either $|G_t/G_p| \geq \aleph_0$ or $p \neq 2$ and $|G_t/G_p| = 2$;
(2) $W_{\alpha,q}(V(FG)) = W_{\alpha,q}(G)$,
for all primes q , provided G is p -mixed.

Proof. (1) Because of the early noted property

$$W_{\alpha,p}(V(FG)) = W_{0,p}(V^{p^\alpha}(FG)) = W_{0,p}(V(FG^{p^\alpha}))$$

(compare with [3]), it suffices to prove the wanted equality only for $\alpha = 0$. By Steinitz theorem (see, e.g., [5]) we have $F \subseteq K$ for some algebraically closed, whence perfect, field K of the same characteristic p which is with the same identity as F .

Assume for a moment that $G_p = 1$. Hence Lemma 2 and the comments in the Remark stated after Proposition 5 apply to deduce that $W_{0,p}(V(FG)) \leq |G_t| \cdot W_{0,p}(G)$. We shall now check the validity of the symmetric inequality $W_{0,p}(V(FG)) \geq |G_t| \cdot W_{0,p}(G)$. To this aim, let $W_{0,p}(G) \neq 0$ (if $W_{0,p}(G) = 0$, [3, Corollary 7] forces that $W_{0,p}(V(FG)) = 0$ and there is nothing to prove), hence there is $g \in G$ with $g \notin G^p$. Choose $e \in FC \setminus \{0, 1\}$ with $e^2 = e$ for some finite subgroup $C \leq G_q$ where $q \neq p$ is a prime. Consider the element $eg + (1 - e)$. Clearly, $eg + (1 - e) \in V(FG) \setminus V(FG^p)$ since $(eg + (1 - e))(eg^{-1} + (1 - e)) = 1$ and if we assume in a way of contradiction that $eg + (1 - e) \in V(FG^p)$, it is plainly seen that $g \in G^p$, because $G_q = G_q^p$ and so $e \in FG^p$, which contradicts our choice and substantiates our claim. Moreover, if e' is another idempotent of FG_l for some prime $l \neq p$; $e' \neq e$ and $e' \in FG^p$, we see that $e'g + (1 - e') \in V(FG) \setminus V(FG^p)$ and $e'g + (1 - e') \neq eg + (1 - e)$ since otherwise $(e - e')g = (e - e')$ and hence $g \in G^p$, because $e - e' \in FG^p$, which is false. On the other hand, we can say even that $(e'g + (1 - e'))V(FG^p) \neq (eg + (1 - e))V(FG^p)$. If not, $(eg + (1 - e))(e'g^{-1} + (1 - e')) \in V(FG^p)$. Multiplying both sides with the idempotent e , we obtain $ee' + e(1 - e')g \in FG^p$. Since both e and ee' lie in FG^p , whence $e(1 - e')$ also lies in FG^p , it is trivial that $g \in G^p$ which is wrong. That is why, the cardinality of $V(FG)/V(FG^p)$ is more than or equal to the number of all different idempotents in FG which number corresponds the number of finite subgroups of G_t . Consequently, $|V(FG)/V(FG^p)| \geq |G_t|$. Likewise, according to ([3, Corollary 3]) the inequality $W_{0,p}(G) \leq W_{0,p}(V(FG))$ holds. Thus, if $W_{0,p}(V(FG)) \geq \aleph_0$, then

$$W_{0,p}(V(FG)) = |V(FG)/V(FG^p)| \geq \max\{|G_t|, W_{0,p}(G)\}$$

which, by what we have already shown above, is tantamount to the desired equality

$$W_{0,p}(V(FG)) = \max\{|G_t|, W_{0,p}(G)\} = |G_t| \cdot W_{0,p}(G).$$

Let us now p is different from 2 and G_t is cyclic of order 2. Take $g \in G^p$ and $h \in G^p$ with $gG^p \neq hG^p$ and e is the idempotent constructed as above. Hence, $(eg + (1 - e))V(RG^p) \neq (eh + (1 - e))V(RG^p)$. Indeed, if the contrary, $(eg + (1 - e))(eh^{-1} + (1 - e)) = egh^{-1} + (1 - e) \in V(RG^p)$, because of $e \in FG^p$, obviously implies

that $gh^{-1} \in G^p$, i.e., $gG^p = hG^p$ which is impossible. Thereby, if $W_{0,p}(V(FG)) < \aleph_0$, then

$$W_{0,p}(V(FG)) = \log_p |V(FG)/V(FG^p)| \geq |G_t| \cdot \log_p |G/G^p| = |G_t| \cdot W_{0,p}(G),$$

since $|V(FG)/V(FG^p)| \geq |G/G^p|^{|G_t|}$, which is precisely the desired inequality.

Suppose now that G_p is arbitrary. It is simple verified that

$$V(FG)/V_p(FG) \cong V(F(G/G_p))$$

and thus, as aforementioned,

$$W_{0,p}(V(FG)) = W_{0,p}(V(FG)/V_p(FG)) = W_{0,p}(V(F(G/G_p))).$$

Henceforth, the previous step works to infer that

$$W_{0,p}(V(FG)) = |(G/G_p)_t| \cdot W_{0,p}(G/G_p) = |G_t/G_p| \cdot W_{0,p}(G)$$

since $(G/G_p)_t = G_t/G_p$ and $W_{0,p}(G/G_p) = W_{0,p}(G)$. Finally, one may deduce that

$$G_t^{p^\alpha}/G_p^{p^\alpha} \cong G_t^{p^\alpha}/(G_t^{p^\alpha} \cap G_p) \cong G_t^{p^\alpha} G_p/G_p = (G_t/G_p)^{p^\alpha} = G_t/G_p$$

since $G_t/G_p \cong \coprod_{q \neq p} G_q$, thus completing this point.

(2) Since G is p -mixed, it is well-known that (see, for Example, [1, 3]) $V(FG)$ is also p -mixed and even more, $V(FG) = GV_p(FG)$. Therefore, $V(FG)/G \cong V_p(FG)/G_p$ is a p -group and thus ([3, Lemma 9]) applies to show that $W_{\alpha,p}(V(FG)) = W_{\alpha,p}(G)$. If now $\alpha \geq \omega$, we employ Lemma 3 to infer that $W_{\alpha,q}(V(FG)) = 0 = W_{\alpha,q}(G)$. But, if $\alpha < \omega$, we subsequently obtain that

$$\begin{aligned} V^{q^\alpha}(FG)/V^{q^{\alpha+1}}(FG) &= G^{q^\alpha} V_p(FG)/G^{q^{\alpha+1}} V_p(FG) \\ &\cong G^{q^\alpha}/(G^{q^\alpha} \cap (G^{q^{\alpha+1}} V_p(FG))) = G^{q^\alpha}/G^{q^{\alpha+1}} \end{aligned}$$

because with the modular law at hand

$$G^{q^\alpha} \cap (G^{q^{\alpha+1}} V_p(FG)) = G^{q^{\alpha+1}} (G^{q^\alpha} \cap V_p(FG)) = G^{q^{\alpha+1}} G_p = G^{q^{\alpha+1}}$$

and $G_p = G_p^q$. Hence $W_{\alpha,q}(V(FG)) = W_{\alpha,q}(G)$ because $V_q(FG) = G_q = 1$. $\square$

Remark. When $|G_t/G_p| \geq \aleph_0$ or $W_{\alpha,p}(G) \geq \aleph_0$, that is, $|G_t/G_p| \cdot W_{\alpha,p}(G) \geq \aleph_0$, we have $W_{\alpha,p}(V(FG)) = \max(|G_t/G_p|, W_{\alpha,p}(G))$ whenever α is an ordinal. In the case when $|G_t/G_p| < \aleph_0$ and $W_{\alpha,p}(G) < \aleph_0$, we have $W_{\alpha,p}(V(FG)) \leq |G_t/G_p| \cdot W_{\alpha,p}(G) < \aleph_0$.

A question which naturally arises is the following:

Problem 2. What are the Warfield p -invariants in the case when F is not necessarily perfect?

What we can say about this at present is that the following claim holds, thus extending ([3, Corollary 7]):

Proposition 7

$W_{0,p}(V(RG)) = 0 \iff W_{0,p}(G) = 0$ and $R = R^p$ when $G_t \neq G_p$, and $W_{0,p}(V(RG)) = 0 \iff W_{0,p}(G) = 0$ when $G_t = G_p$, provided that R is a commutative unitary ring of prime characteristic p without nilpotent elements and also without idempotents in the second case.

Proof. In fact, if $G_t = G_p$ then as above $V(RG) = GV_p(RG)$ and therefore if $G = G^p G_p$ then

$$V(RG) = G^p V_p(RG) \subseteq V(R^p G^p) V_p(RG),$$

i.e., $V(RG) = V(R^p G^p) V_p(RG) = V^p(RG) V_p(RG)$ without $R = R^p$ (compare also with the [3, proof of Theorem 10]).

Let now $G_t \neq G_p$ and $W_{0,p}(V(RG)) = 0$. Hence there exists $e = (1/q) \sum_{1 \leq i \leq q} b^i$ where $\langle b \rangle \subseteq G_q$ is cyclic of order q for some prime $q \neq p$; so q inverts in R . It is easily checked that $e^2 = e$. Take $r \in R \setminus \{0, 1\}$; thus $r^{p^k} \neq \{0, 1\}$ for all integers k . It is also a plain exercise to verify that $er + (1 - e) \in V(RG) = V(R^p G^p) V_p(RG)$. That is why, there is $m \in \mathbb{N}$ such that $er^{p^m} + (1 - e) \in V(R^{p^{m+1}} G^{p^{m+1}})$. Consequently,

$$\begin{aligned} \sum_{1 \leq i \leq q} (q^{-1} r^{p^m} - q^{-1}) b^i + 1 &= \sum_{1 \leq i \leq q-1} (q^{-1} r^{p^m} - q^{-1}) b^i \\ &+ q^{-1} r^{p^m} - q^{-1} + 1 \in V(R^{p^{m+1}} G^{p^{m+1}}). \end{aligned}$$

Since $b^i \neq 1$ for each $1 \leq i \leq q-1$, the canonical record yields $q^{-1} r^{p^m} - q^{-1} \in R^{p^{m+1}}$. But $q \in R^{p^{m+1}}$, whence $q^{-1} \in R^{p^{m+1}}$. Furthermore, $r^{p^m} \in R^{p^{m+1}}$ and the lack of nilpotent elements ensures that $r \in R^p$. So, $R = R^p$. That $W_{0,p}(G) = 0$ follows from ([3, Corollary 3]).

Conversely, let us assume that $R = R^p$ and $G = G^p G_p$. We shall essentially follow the idea from ([3, Proposition 6]). Indeed, given $x \in V(RG)$, hence $x = r_1 g_1 + \dots + r_s g_s$ with $r_i \in R$, $\sum_{1 \leq i \leq s} r_i = 1$ and $g_i \in G$, $1 \leq i \leq s \in \mathbb{N}$. Write $r_i = f_i^p$ for some $f_i \in R$ and $g_i = a_i^p c_i$ for some $a_i \in G$ and $c_i \in G_p$. Furthermore,

$$\begin{aligned} x &= f_1^p a_1^p c_1 + \dots + f_s^p a_s^p c_s = 1 + f_1^p (a_1^p c_1 - 1) + \dots + f_s^p (a_s^p c_s - 1) \\ &= 1 + f_1^p (a_1^p (c_1 - 1) + a_1^p - 1) + \dots + f_s^p (a_s^p (c_s - 1) + a_s^p - 1) \\ &= 1 + f_1^p a_1^p (c_1 - 1) + f_1^p (a_1^p - 1) + \dots + f_s^p a_s^p (c_s - 1) + f_s^p (a_s^p - 1) \\ &= 1 + f_1^p (a_1^p - 1) + \dots + f_s^p (a_s^p - 1) + f_1^p a_1^p (c_1 - 1) + \dots + f_s^p a_s^p (c_s - 1). \end{aligned}$$

Since there is $t \in \mathbb{N}$ such that $c_i^{p^t} = 1$ for all $1 \leq i \leq s$, we deduce that

$$x^{p^t} = [1 + f_1^p (a_1^p - 1) + \dots + f_s^p (a_s^p - 1)]^{p^t}.$$

By setting $u = 1 + f_1^p (a_1^p - 1) + \dots + f_s^p (a_s^p - 1)$, we elementarily infer that $u \in V(R^p G^p)$ since $x \in V(RG)$. Finally, one can decompose

$$x = u(1 + f_1^p a_1^p u^{-1} (c_1 - 1) + \dots + f_s^p a_s^p u^{-1} (c_s - 1)) = uv,$$

where it is simple checked that

$$v = 1 + f_1^p a_1^p u^{-1} (c_1 - 1) + \dots + f_s^p a_s^p u^{-1} (c_s - 1) \in V_p(RG).$$

Thereby, the desired decomposition $V(RG) = V(R^p G^p)V_p(RG) = V^p(RG)V_p(RG)$ holds, as expected. $\square$

By analogy with the relative Ulm-Kaplansky invariants (e.g., [4], v. II), we define for each ordinal α the relative Warfield q -invariants of the group G with respect to its subgroup A like this (compare with [2] as well):

$$W_{\alpha,q}(G, A) = \text{rank}(G^{q^\alpha} / [(G^{q^{\alpha+1}} A) \cap G^{q^\alpha}] G_q^{q^\alpha}) = \text{rank}(G^{q^\alpha} / [(G^{q^{\alpha+1}} A) G_q^{q^\alpha}] \cap G^{q^\alpha}).$$

It is straightforward that $W_{\alpha,q}(G, G^{q^\rho}) = W_{\alpha,q}(G)$ for every ordinal number $\rho > \alpha$. A problem of interest is the following:

Problem 3. Compute $W_{\alpha,q}(V(RG), U)$ for certain subgroups U of $V(RG)$.

However, this is the theme of some new research exploration.

Now, we come to an isomorphism assertion pertaining to the inheritance of $W_{\alpha,p}(G)$ from the F -algebra FG . Specifically, the following statement is true.

Proposition 8

Let F be a field of characteristic $p > 0$ and G an Abelian group. Then, for each ordinal α , $FG \cong FH$ as F -algebras for some group H implies that $W_{\alpha,p}(G) = W_{\alpha,p}(H)$ in the cases when $W_{\alpha,p}(G) \geq |G_t/G_p| \geq \aleph_0$ or $|G_t/G_p| < \aleph_0$.

In particular, if G is p -mixed, $FG \cong FH$ as F -algebras implies $W_{\alpha,p}(G) = W_{\alpha,p}(H)$.

Proof. By virtue of [6], without loss of generality, we may assume that F is algebraically closed, whence perfect. Utilizing Proposition 5 and the last part of point (1) from Theorem 6, we obtain that either $W_{\alpha,p}(G) = W_{\alpha,p}(V(FG))$ or $W_{\alpha,p}(G) = W_{\alpha,p}(V(FG))/|G_t/G_p|$.

On the other hand, observe that $V(FG)$ can be retrieved from FG , so the same holds for $W_{\alpha,p}(V(FG))$. Moreover, the isomorphism between FG and FH yields an isomorphism between $F(G/G_p)$ and $F(H/H_p)$ (see, e.g., [6]). Therefore, owing to [7], we have $|(G/G_p)_t| = |(H/H_p)_t|$, i.e., $|G_t/G_p| = |H_t/H_p|$. This forces the wanted equality between $W_{\alpha,p}(G)$ and $W_{\alpha,p}(H)$. $\square$

Finally, we ask the following:

Problem 4. Whether or not $W_{\alpha,p}(G, A)$ may be determined by the pair of F -algebras (FG, FA) ?

This is, however, a theme of some other research study.

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