Abū Sahl al-Kūhī's "On Drawing Two Lines from a Point at a Known Angle"

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Introductory Remarks

Abū Sahl al-Kūhī, a late tenth-century Iranian mathematician with some thirty works to his credit, was an expert geometer in the Greek tradition who worked mostly in Baghdad for several members of the ruling Būyid family.[1] His works deal with virtually every geometric aspect of the studies of the great Greek mathematicians, including those of Euclid, Archimedes, and Apollonius. Although he ventured occasionally to apparently astronomical problems (such as the distance to the shooting stars[2] and the astrolabe[3]), his approach and interests in even these topics were none-theless purely mathematical. The present work[4] is remarkable

1 For a biography of al-Kūhī, see Yvonne Dold-Samplonius 1975.

2 See Berggren and Van Brummelen 2001.

3 See Berggren 1994 for an edition and English translation and Rashed 1993, pp. 190-230 for an edition and French translation.

4 The treatise is listed in Sezgin 1974 as the eighth entry under al-Kūhī, p. 319. It appears that the first notice of the contents of this treatise appears to have been given by Woepcke (1851, pp. 55-56). Woepcke, however, identified it with a treatise

not only for its mathematical content, but also for its exclusive use of the ancient method of geometric analysis,[5] and its successive reductions of new cases of the problem to previously solved cases.

The general problem addressed here is as follows: given a line (or circle) and a point not on it, draw two lines from the point, containing a given angle and intersecting the given line (or circle) in two points such that some condition holds. This collection of related problems has the feel of some of Apollonius's minor works, i.e., collections of auxiliary tools that might be useful in solving other problems. Thus al-Kūhī might have felt that this work was contributing to laying the foundation for future geometrical research. The original motivation may also have come, in part, from al-Kūhī's treatise on the construction of the astrolabe,[6] for in the final section of the second book of that work al-Kūhī quotes Problems 8 and 11 of "On Drawing Two Lines" in order to justify certain steps taken earlier. The analysis of Problem 8 in the astrolabe treatise is identical to that found in the present work, but the analysis of Problem 11 is slightly different since al-Kūhī is unable to rely on the reduction he used in "On Drawing Two Lines" without quoting two other problems and their (reductive) solutions.

We also find a problem (the second) from this work in a treatise by al-Kūhī's colleague al-Sijzī, *Book on Making Easy the Ways of Deriving Geometrical Figures*,[7] where it is given as an example of a problem in the art of analysis that may be attempted by a beginner. Al-Sijzī's analysis is different from al-Kūhī's, again because of the use of reduction in al-

mentioned in the *Fihrist* on producing points on lines in the ratio of areas (Sezgin 1974, #29, p. 321), but this is certainly a different treatise.

5 We have recently published two of al-Kūhī's works which apply the full Islamic expression of analysis and its usual partner synthesis; see Berggren and Van Brummelen 1999/2000 and 2000 [1].

6 See Berggren 1994. The text of the relevant section may be found at pp. 178-179; relevant commentary is on pp. 196-197 and 202-203.

7 Hogendijk and Bagheri 1996, 29-30.

Kūhī's work. Perhaps, then, the problem had some currency in its own right.

Analysis, as understood by the Greeks, was a collection of techniques that were useful in finding paths to solutions of problems. Pappus, in his discussion of analysis in Book vii of his *Mathematical Collection*, distinguishes between *theoretical analysis*, which is used to find a proof of a theorem, and *problematical analysis*, which aids in solving a construction problem. The latter was by far the most popular, both in Greek and Islamic times, and it is what we find here.

Pappus describes problematical analysis as follows: "we posit the [problem] set forth as known and then pass through the things which follow in order [from it], [positing them] as true, to something admitted. If the thing admitted is possible and obtainable, what they call in mathematics given, the [problem] set forth also will be possible, and again the proof will be the reverse of the analysis; but if we should happen upon something admitted to be impossible, the problem also will be impossible."

A cursory observation of this treatise demonstrates that the key issue in understanding problematical analysis is the meaning of the term "given" — in Arabic geometry, "known". That this is no trivial matter is witnessed by debates on the matter both in ancient Greece and in medieval Islam. The standard tool for use in analysis is Euclid's *Data*,[8] a collection of results which state that if certain aspects of a geometrical configuration are given, then other aspects are also given. Marinus of Neapolis' commentary on the *Data*[9] gives three competing definitions of "given": (a) the object is constructible by Euclidean methods; (b) the object is determined to one or finitely many possible configurations; (c) the object (or its construction) is accessible to human reason, or comprehensible. While (a) accords with the apparent correlation between propositions in the *Data* and the *Elements* and between the standard practices of analysis

8 This treatise is available, for instance, in Ito 1980.

9 Heiberg and Menge 1896, p. 234.

and synthesis, there are some difficulties with it.[10] Marinus ascribes (b) to Apollonius, and it is essentially al-Kūhī's position as outlined in his correspondence with al-Ṣābī.[11]

It is worth noting that al-Kūhī makes extensive and thorough use of the *Data* throughout (although the references are never explicit). In fact, he might have seen this work in part as an exercise to demonstrate the use of the *Data* to solve geometric problems. While he uses propositions from almost every part of the *Data*, he never goes beyond its scope, not even to use his own treatise on supplementary results to the *Data*.

The treatise shows signs of being completed in some haste. Although the determination of conditions for possibility of solution (*diorismoi*) is fundamental to analysis and the problems posed here are rife with such conditions, al-Kūhī does not address them at all. He refers to *diorismoi* only in a brief note at the end of the work, when he refers to several variations of some of the problems. Also, al-Kūhī seems to have been so confident that his analyses solved the problems that he did not even bother to supply syntheses. This is in opposition to the standard practice, wherein the syntheses are given but the analyses often suppressed.

As we said earlier, al-Kūhī makes extensive use of reduction of one problem to one that has been solved previously throughout the work. Of the eleven problems, only Problems 1 and 8 are solved directly. These correspond to the first instances of the two different types of problem (where the given object is a line or circle respectively). In many cases it is possible to do the analysis directly just as easily as al-Kūhī's reduction; perhaps then he was motivated by some sort of aesthetic consideration or a sense of play. This style of analysis does not lend itself to easy syntheses, and one wonders how seriously he took his own statement, at the end of the work, that he hoped to finish them someday.

10 For an exposition of these matters, see Berggren and Van Brummelen 2000 [2].

11 See Berggren 1983, 39-124, especially pp. 52-56.

Translation

Our edition of the text is based on the unique known copy of this work (Bibliothèque Nationale Fonds Arabe 2457, ff. 48 – 50); the copyist was al-Sijzī, who was well-acquainted with al-Kūhī's work.[12] The reader will note that although we use the spelling 'al-Kūhī' elsewhere in the paper we have, in the translation of the manuscript, used the spelling 'al-Qūhī'. We do so because, although both spellings of the name are documented, this is the form that appears in this manuscript. Our system for transliterating letters referring to points in the diagrams is that of Kennedy 1991/92. In our edition of the text we have used superscript Arabic numerals to refer to our critical notes. In editing the text we have not noted the few places in which we have changed the orthography to standard classical orthography, nor the several places where we have corrected what seemed to be trivial scribal errors, such as misreadings of *yā'* for *tā'* (and conversely). Our policy on noting what could be construed as scribal errors in copying letters referring to points in geometrical diagrams is that if the form of the letter in the text could be construed as representing the correct letter then we took it as being correct. Otherwise we made a note.

In the name of God, the Merciful, the Compassionate

The Treatise on Drawing Two Lines from a Point at a Known Angle by the Method of Analysis
Wījan b. Rustam, known as Abū Sahl al-Qūhī, solved it

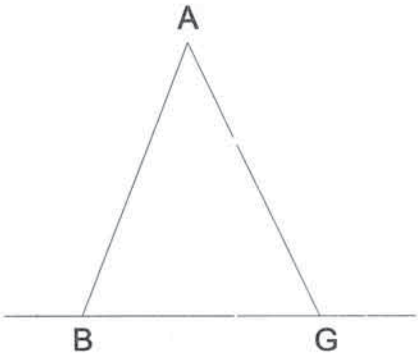
He said: Our purpose in this treatise is to draw two straight lines from a known point at a known angle to a line known in position so that the ratio

[12] Although al-Sijzī's role as the copyist of the treatises in the codex has been disputed, Paul Kunitzsch and Richard Lorch 1993 review the controversy and present convincing evidence that al-Sijzī did indeed copy all but a few folios at the end of this important collection.

of one of the two of them to the other, or a rectangle[13], or the magnitude of the resulting triangle, or its base, or their two squares together, or the two of them together, or the difference between them is known.

So let the known point be the point A, and the line known in position be BG, and suppose first of all that it is straight.

[Problem 1] We want to draw from the point A two straight lines to the line BG, containing a known angle, so that the ratio of one of the two of them to the other is known.



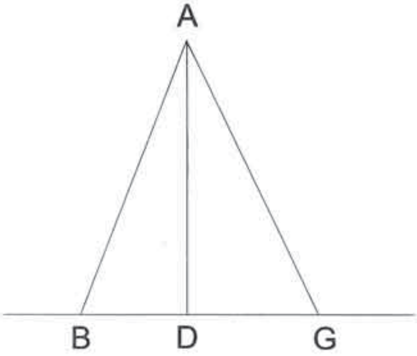
Analytically, we suppose that the two lines drawn are AB and AG, that the ratio of AB to AG is known, and angle BAG is known. Then, since the ratio of the line BA to AG is known, and angle BAG is known, triangle ABG is known in form. Therefore angle ABG is known; but point A is known, so line AB is known in position. Therefore line AG is known in position, since angle BAG is known. And line BG is known in position, and therefore each of the two points B and G is known,[14] and that is what we wanted to know.

[Problem 2] We want to draw from the known point A two straight lines to the straight line known in position, BG, which contain a known angle, so that the rectangle of one of them by the other is known.

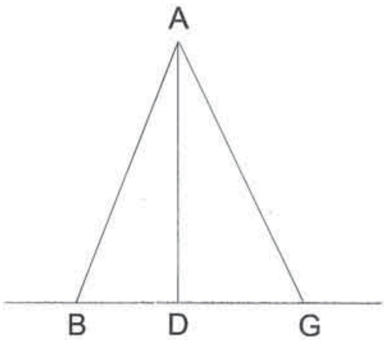
[13] The Arabic here is "al-saṭḥ", which denotes any sort of plane area. But, since the reference is clearly to an area in the form of a rectangle we shall simply use "rectangle" here and throughout the treatise.

[14] One might wonder why al-Kūhī did not stop once he had determined the two lines AB and AG as known. All we can suggest is that he regarded the problem as one of producing segments in a certain ratio, so he wanted to produce not just the lines but the segments that these lines, together with BG, created.

Analytically, we suppose that the two lines drawn are BA and AG, that they contain the known angle BAG, and that the rectangle BA by AG is known. But the ratio of rectangle BA by AG to triangle BAG is known, since angle BAG is known, and therefore the triangle BAG is known in magnitude. And the ratio of the triangle ABG to the rectangle BG by the perpendicular AD drawn from the point A to the line BG is known.



Therefore the rectangle BG by AD is known in magnitude. And line AD is known in magnitude, so line BG is known in magnitude. Therefore its square is known, and so the ratio of the square of BG to the rectangle contained by BA and AG is known. And angle BAG is known, so triangle ABG is known in form. Therefore, the ratio of BA to AG is known. And angle BAG is known, and each of the two points B, G is known, as we showed in the preceding figure,[15] and that is what we wanted to know.



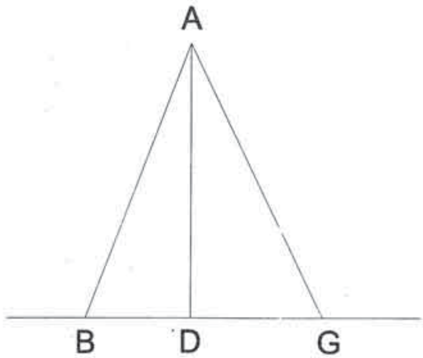
[Problem 3] We want to draw from the known point A to the straight line BG, which is known in position, two straight lines containing a known angle so that the resulting triangle is known in magnitude.

Analytically, we suppose that the two lines drawn are AB, AG, that they contain the known angle BAG, and that triangle ABG is known in magnitude. But the ratio of triangle ABG to the rectangle BA by

[15] i.e., 'preceding problem.'

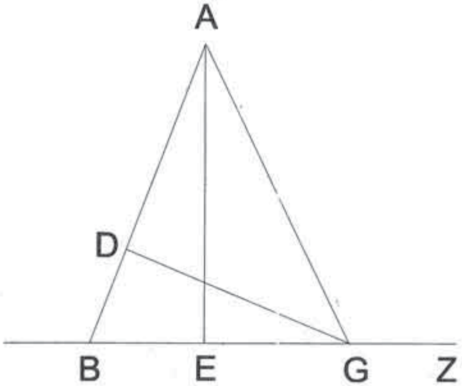
AG is known, since angle BAG is known, so the rectangle BA by AG is known in magnitude. And angle BAG is known, so each of the points B and G is known, as we proved in the preceding case. And that is what we wanted to know.

[Problem 4] We want to draw from the known point A to the straight line BG, which is known in position, two straight lines containing a known angle so that the base of triangle ABG, namely BG, is known.



Analytically, we suppose that the two lines drawn are AB and AG, that they contain the known angle BAG, and that the base BG is known. So the rectangle contained by BG and the perpendicular from the point A to the base BG, namely AD, is known. But the ratio of the rectangle BG by AD to the triangle ABG is known, so the triangle ABG is known in magnitude. Thus the two points B and G are known, as we proved in the preceding case, which is what we wanted to know.

[Problem 5] We want to draw from the known point A two straight lines to the straight line BG, which is known in position, containing a known angle so that the sum of the squares of the two of them is equal to a known rectangle.



Analytically, we suppose: the two lines drawn are BA and AG; they contain the known angle BAG; the sum of the squares of the two lines BA and AG is known; and line GD is perpendicular to line AB. Then, since the sum of the squares of the two lines BA and

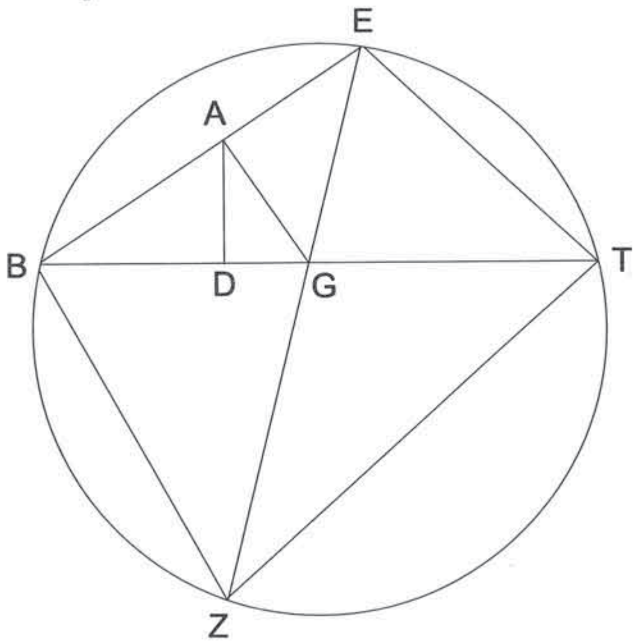
AG is equal to the sum of the square of line BG and twice the rectangle BA by AD, the sum of the square of line BG and twice the rectangle BA by AD is known. Also, since angle DAG is known, and angle GDA is right, triangle ADG is known in form.

Thus, the ratio of line GD to DA is known, and therefore if we make this ratio as the ratio of AE, the perpendicular from point A to line BG, to another line, say GZ, the line GZ is known in magnitude, since the line AE is known in magnitude, and since the ratio of the line AE to line GZ is as the ratio of the line GD to line DA. And the ratio of line GD to DA is as the ratio of the rectangle GD by AB to the rectangle DA by AB, since AB is the height common to both of them. And the rectangle GD by AB is equal to the rectangle AE by BG, since each of them is twice triangle ABG. Thus the ratio of rectangle AE by BG to the rectangle DA by AB is as the ratio of line AE to GZ.

But the ratio of line AE to GZ is as the ratio of the rectangle AE by BG to the rectangle GZ by GB, since BG is the height common to both of them, and so the ratio of the rectangle AE by BG to each of the two rectangles BG by GZ and BA by AD is the same. Therefore the rectangle BA by AD is equal to the rectangle BG by GZ, and so double the rectangle BA by AD is equal to double the rectangle BG by GZ. Thus the sum of the square of line BG and double the rectangle BG by GZ is known. And we add to it the square of line GZ, which is known. Therefore the totality will be the square BZ, and so the square of the line BZ is known. Thus line BZ is known in magnitude. But line GZ is known in magnitude, so BG, the remainder, is known in magnitude. And angle BAG is known, so each of the two points B, G is known, according to what we proved in the preceding case, and that is what we wanted to know.

[Problem 6] We want to draw from the known point A two straight lines to the straight line BG, which is known in position, containing a known angle so that the sum of the two of them is known in magnitude.

Analytically, we suppose that the two lines drawn are AB and AG, that they contain the known angle BAG, and that the sum of the two lines BA, AG is known in magnitude. And the ratio of the sum of the two lines BA and AG to the perpendicular drawn from point A to line BG, say AD, is [therefore] known.

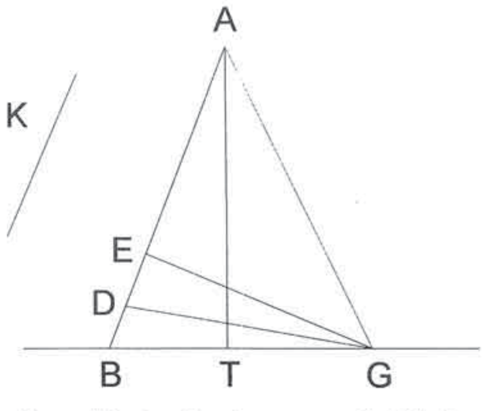


Therefore if we imagine line AE equal to line AG, and line BAE straight, the ratio of line EB to line AD is known. And we make line BG a common height for the two of them, and therefore, the ratio of the rectangle EB by BG to the rectangle AD by BG is known. And the ratio of the rectangle AD by BG to triangle ABG is known, since it is its double. And the ratio of the triangle ABG to the rectangle contained by GA and AB is known, since angle BAG is known, so the ratio of the rectangle EB by BG to the rectangle GA by AB, i.e. to the rectangle EA by AB, is known.

And therefore, if we imagine line BZ parallel to line AG, and line EGZ straight, and the rectangle contained by EG and GZ equal to the rectangle contained by BG and GT, the triangle EBZ will be isosceles, since it is similar to triangle EAG. (And triangle EAG is known in form, since angle EAG is known.) And so the ratio of the rectangle EA by AB to the rectangle EG by GZ is known, since it is as the ratio of the square of EA to the square of EG. And so the ratio of the rectangle EB by BG to the rectangle EG by GZ is known. And the rectangle EG by GZ is equal to the rectangle BG by BT, and so the ratio of the rectangle EB by BG to the rectangle BG by GT is known. And it is as the ratio of EB, which is known, to the line GT, since BG is a common height to the two of them; so the line GT is known.

And also, since the rectangle EG by GZ is equal to the rectangle BG by GT, we may describe through the points E, B, Z and T a circle. And we draw the two lines ET and TZ. Then angle ETB is equal to angle ZTB, since the chord EB is equal to the chord BZ, and angle BTZ is equal to angle ZEB, since their bases are the same arc. Therefore angle BTE is equal to angle GEB, and angle EBG is common; hence, triangle EBT is similar to triangle EBG. And so the ratio of the line TB to BE is as the ratio of line EB to BG and thus the rectangle TB by BG is equal to the square of line EB. But the square of line EB is known, since EB is known, and so the rectangle TB by BG is known, and line TG was shown [above] to be known. Therefore, line BG is known, and so its ratio to the sum of the two lines BA and AG is known. And angle BAG is known, so triangle ABG is known in form. Therefore, the ratio of BA to AG is known, and so each of the two points B, G is known [by the first case], and that is what we wanted to know.

[Problem 7] We want to draw from the known point A two straight lines to the straight line BG, which is known in position, containing a known angle so that their difference is known.



Analytically, we suppose that the two lines drawn are BA, AG and that they contain the known angle BAG, and that the excess of line BA over AG, say BD, is known. Therefore the remaining line, DA, is equal to line AG. Therefore, if we imagine line GE to be perpendicular to AB, then the square of line BG is, together with double the rectangle BA by AE, equal to the [sum of the] squares of the two lines BA, AG. But the [sum of the] squares of the two lines BA, AG is equal to the [sum of the] squares of the two lines BA, AD, since AD is equal to line AG, and so the square of line BG, with double the rectangle BA by AE, is equal to the [sum of the] squares of the two lines BA, AD.

And the [sum of the] squares of the two lines BA, AD is equal to double the rectangle BA by AD with the square of the line BD, and so the square of the line BG with double the rectangle BA by AE is equal to the square of the line BD with double the rectangle BA by AD. But double the rectangle BA by AD is equal to double the rectangle BA by AE with double the rectangle BA by ED, and so the square of the line BG with double the rectangle BA by AE is equal to the square of the line BD with double the rectangle BA by AE and double the rectangle BA by DE. Therefore if we remove double the rectangle BA by AE, which is common, there remains the square of BG equal to the square of BD with double the rectangle BA by ED. And if we imagine the rectangle GB by BZ equal to the square of BD, the rectangle GB by BZ with double the rectangle BA by DE is equal to the square of the line BG. But the square of the line BG is equal to two rectangles, one of them GB by BZ and the other BG by GZ. And so if we remove the rectangle GB by BZ, which is common, there remains the rectangle BG by GZ equal to double the rectangle BA by DE. And so the rectangle BA by DE is equal to the rectangle BG by half the line GZ.

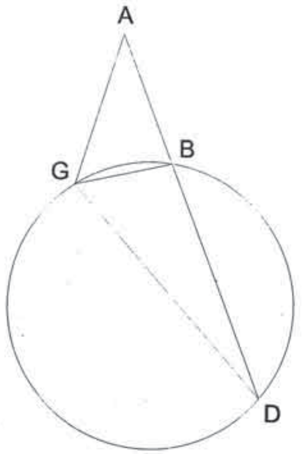
And also, since angle DEG is right, and angle GDE is known (so triangle GED is known in form), the ratio of the line GE to ED is known. And so if we make the ratio of the line AT, the perpendicular from point A to line BG, to another line, say K, as the ratio of the line GE to line ED, the line K is known, since line AT is known and since the ratio of the line AT to line K is as the ratio of line GE to line ED.

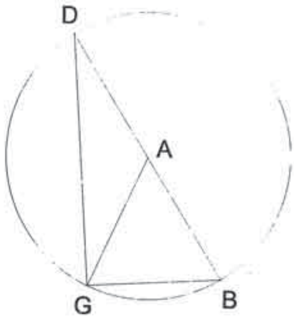
But the ratio of line GE to line ED is as the ratio of the rectangle AB by GE to [the rectangle] AB by ED. Also, the ratio of the line AT to K is as the ratio of the rectangle BG by AT to the rectangle BG by K, and so the ratio of the rectangle BA by GE to the rectangle BA by DE is as the ratio of the rectangle BG by AT to the rectangle BG by K. And the rectangle BA by GE is equal to the rectangle BG by AT, since each of them is equal to double the triangle ABG, and so the rectangle BA by ED is equal to the rectangle BG by K. And it was proved that the rectangle BA by ED is equal to the rectangle BG by half the line ZG, and so the rectangle BG by half the line ZG is equal to the rectangle BG by K.

Therefore, line K is equal to half the line ZG, and so line ZG is known, since K, which was half of it, is known. And the rectangle GB by BZ is known, since it is equal to the square of the known line BD, and the line BZ is known, and so all of BG is known, and line AT is known, and so the rectangle one of them by the other is known, and it is double triangle ABG. And so triangle ABG is known, and angle BAG is known, and each of the two points B, G is known, as we proved in the figure which preceded, and that is what we wanted to show.

And let the line known in position, which is BG, be part of a circumference of a circle known,in position.

[Problem 8] We want to draw from the known point A two straight lines, which contain a known angle, to the line BG, which is [part of] the circumference of a circle, so that the ratio of one of them to the other is known.





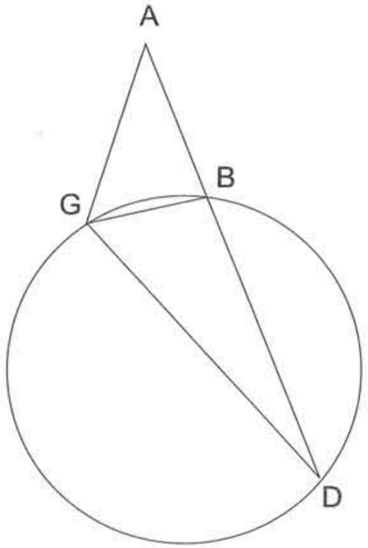
Analytically, we suppose that the two lines drawn are AB, AG, that the angle BAG is known, and that the ratio of BA to AG is known. And we draw the line BG, so the rectilineal triangle ABG is known in form; therefore, angle B is known. Thus, if we imagine that line BAD[16] is straight, the chord of arc D[B]G, namely the line DG, is known, since angle B on the circumference of a circle known in position is known. Therefore the square of the line DG is known.

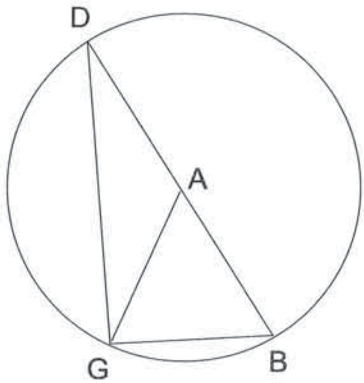
Also, because the ratio of the line BA to AG is known, if AD is made a common height to the two of them, the ratio of the rectangle BA by AD to the rectangle GA by AD is known. But the rectangle BA by AD is known, since point A is known, and therefore the rectangle GA by AD is known. And therefore the ratio of [the rectangle] GA by AD to the square of GD is known. And angle GAD is known. Therefore, triangle GAD is known in form, and so the ratio of the line AG to the line GD is known. But line GD is known, and therefore the line AG is known. And point A is known, so the line AG is known in position. And the circumference of the circle is known in position and so point G is known. And since angle GAB is known the point B is also known. And so each of the two points G and B is known. And that is what we wanted to know.

[Problem 9] We want to draw from point A, which is known, two straight lines, containing a known angle, to the line BG, which is part of the circumference of a circle known in position, so that the rectangle of one of them by the other is known.

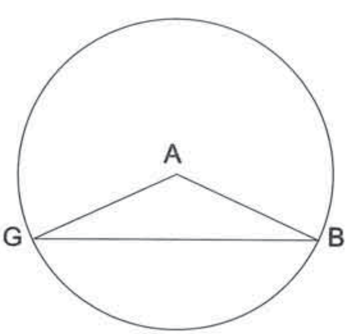
Analytically, we suppose that the two lines drawn are BA, AG, which contain the known angle BAG, and the rectangle BA by AG is known. Also the rectangle BA and AD is known, since the point A is known. Therefore the ratio of the rectangle BA by AG to the rectangle BA by AD is known, and it is as the ratio of the line GA to AD, since the line BA is a common height to both of them. Therefore the ratio of the line GA to AD is known, and angle GAD is known, so the point G is known, as we proved in the preceding figure. Therefore point B is known, since angle GAB is known. And that is what we wanted to know.

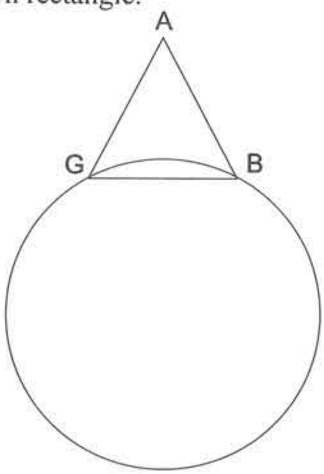
16 Presumably this refers to the diagram with A inside of the circle.





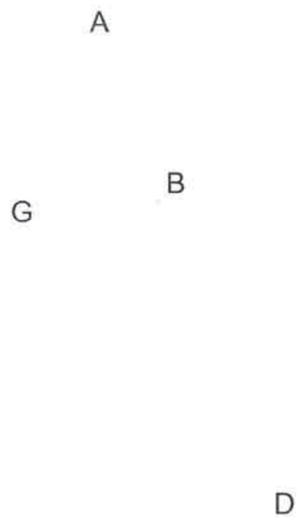
[Problem 10] We want to draw from the known point A two straight lines, containing a known angle, to the line BG known in position, which is part of the circumference of a circle known in position, so that the rectilineal triangle ABG is equal to a known rectangle.



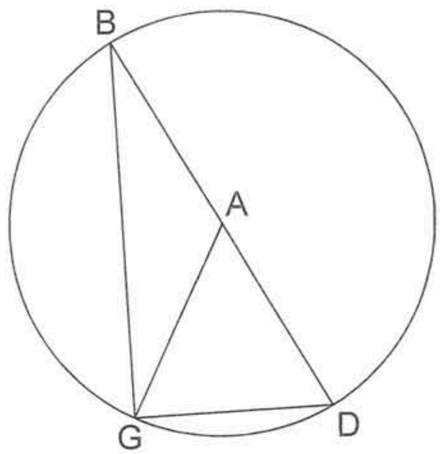


Analytically, we suppose that the two lines drawn are BA and AG, containing the known angle BAG, and that triangle BAG is known in magnitude. But, the ratio of triangle ABG to the rectangle BA by AG is known, since angle BAG is known. And therefore the rectangle BA by AG is known in magnitude. And angle BAG is known, so each of the two points B, G is known according to what we proved in the preceding figure. And that is what we wanted to know.

[Problem 11] We want to draw from the known point A two straight lines, containing a known angle, to the line BG, which is part of the circumference of a circle known in position, so that the base of the triangle, BG, is known.



Analytically, we suppose that the two straight lines drawn are BA and AG, that they contain the known angle BAG, and that BG is known. Therefore we produce line BA in a straight line to D, and we draw DG. Therefore, since BG is known, angle BDG is known. And angle DAG is known, and so triangle DAG is known in form. Therefore the ratio of DA to AG is known, which is as the ratio of the rectangle DA by AB to the



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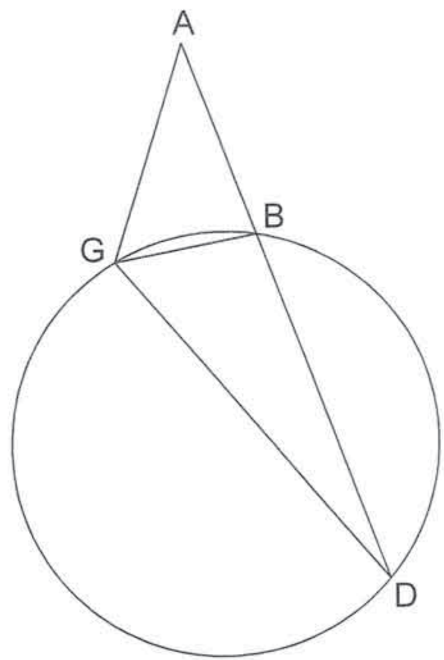


rectangle GA by AB. But the rectangle DA by AB is known, and so the rectangle GA by AB is known. And angle BAG is known, so the two points B, G are known, as we proved in the preceding figure. And that is what we wanted to know.

And when the known point is on the circumference of the circle finding the two points at which the two lines drawn terminate [according to any of the above four criteria] is even easier and clearer. And because it is so obvious we will omit it. But when the known point is the centre of the circle the question is either possible and indeterminate, or impossible. And if the known point is not the centre of the circle, then there are various cases for the point which is to be found, and we omit mention of them because they involve notions of *diorismos*. And if we were to go into partition [into cases], *diorismos*, synthesis and the positions of the points by Apollonius's method in some of his books a very big book would be produced, but we hope to have the leisure to do it later, God willing.

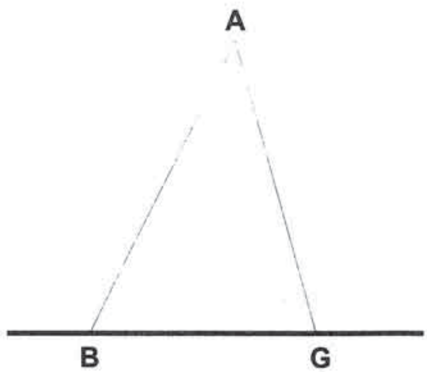
Praise to God, and His blessings on Muḥammad and his family, and peace.

It was collated with the [i.e., al-Kūhī's] original [presumably by al-Sijzī].

Commentary

The general problem is to draw from a point A to a line or circle BG two line segments containing a given angle, so that some condition holds. The first seven of the 11 propositions deal with the case when BG is a line, the last four when BG is a circle.

Problem 1: Draw two line segments from A to the line BG, containing a given angle, so that $\frac{AB}{AG}$ is equal to a given ratio.

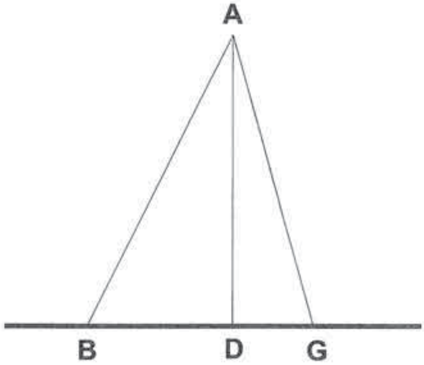


Analysis:

Since $\frac{AB}{AG}$ and $\angle BAG$ are given,

ΔABG is known in form (*Data*, 41); thus $\angle ABG$ is known.
But point A is known; therefore line AB is known in position (*Data*, 30); and since $\angle BAG$ is given, line AG is also known in position (*Data*, 29). Since, then, line BG is known in position, points B and G are both known (*Data*, 25).

Problem 2: Draw two line segments from A to the line BG, containing a given angle, so that $AB \cdot AG$ is equal to a given area.



Analysis:

Since $\angle BAG$ is given, $\dfrac{BA \cdot AG}{\Delta ABG}$ is known (*Data*, 66); but $BA \cdot AG$ is given, so ΔABG is known in magnitude (*Data*, 2).

Draw AD perpendicular to BG (def. D); then $\dfrac{\Delta ABG}{BG \cdot AD}\left(=\dfrac{1}{2}\right)$ is known, which implies that $BG \cdot AD$ is known (*Data*, 2). But AD is known in magnitude,[17] hence BG is known in magnitude (*Data*, 57).

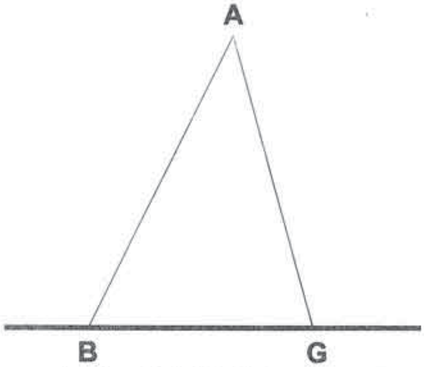
Thus BG^2 is known in magnitude (*Data*, 52); thus $\dfrac{BG^2}{BA \cdot AG}$ is known (*Data*, 1). And $\angle BAG$ is given; hence ΔABG is known in form (*Data*, 80).

Therefore $\dfrac{BA}{AG}$ is known, which reduces the problem to Problem 1.

[17] This can be shown several ways, for instance, by using *Data*, 28 to construct a parallel to BG through A and then applying *Data*, 32; or, by applying *Data*, 30 to show that AD is known in position, hence D is known (*Data*, 25), hence AD is known in magnitude (*Data*, 26).

This proof highlights the sometimes delicate nature of an argument by analysis. Once al-Kūhī determines that ΔABG is known in magnitude, it is easy to presume (incorrectly) that he has completed an analysis of Problem 3. In fact, he has achieved only a reduction of Problem 2 to Problem 3 — which does not help, since he is about to solve Problem 3 by reducing to Problem 2. Had al-Kūhī completed the analysis from this point without using the assumption that the magnitude of quantity $AB \cdot AG$ is given from the statement of Problem 2, he would have completed an analysis for Problem 3 as well; but he does use this assumption about $AB \cdot AG$ later. (An identical situation arises here with respect to Problem 4.) Thus al-Kūhī must proceed to solve Problems 3 and 4, which are in any case reduced easily to Problems 2 and 3 respectively.

Problem 3: Draw two line segments from A to the line BG, containing a given angle, so that ΔABG is equal to a given magnitude.



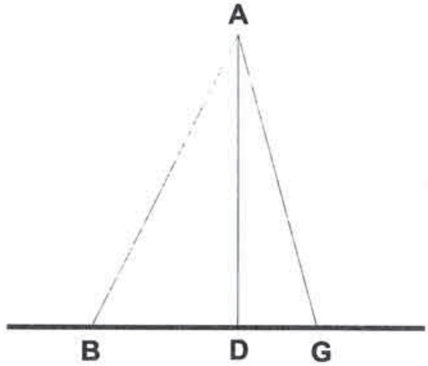
Analysis:

Since $\angle BAG$ is given, $\dfrac{\Delta ABG}{BA \cdot AG}$ is known (*Data*, 66); hence $BA \cdot AG$ is known (since the magnitude of ΔABG is given, *Data*, 2). This reduces the problem to Problem 2.

Problem 4: Draw two line segments from A to the line BG, containing a given angle, so that BG is equal to a given magnitude.

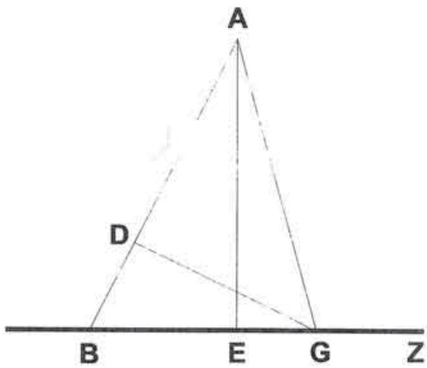
Analysis: Draw AD perpendicular to BG (def. D); then $BG \cdot AD$ is known.[18]

But $\dfrac{BG \cdot AD}{\Delta ABG}\left(=\dfrac{2}{1}\right)$ is known, so ΔABG is known in magnitude (*Data*, 2). This reduces the problem to Problem 3.



Problem 5: Draw two line segments from A to the line BG, containing a given angle, so that $AB^2 + AG^2$ is equal to a given area.

Analysis:
Draw GD perpendicular to AB (def. D); then $BG^2 + 2BA \cdot AD$ $(= AB^2 + AG^2$ by *Elements*, II.13[19]) is known.



[18] An obvious way to see this is to apply *Data*, 76 to find that AD/BG is known; thus the rectangle formed by BG and AD is known in form, and since BG is given in magnitude, the result follows by *Data*, 52.

[19] *Elements* II.13 requires that the triangle contains only acute angles, but many commentators have noted that this is not necessary. For II.13 to hold the given angle BAG must be acute; otherwise D does not lie between A and B and the proposition is false. If angle ABG is obtuse, then D does not lie between A and B either, but the proposition still holds. In any case, if the angle at B is obtuse then the angle at G must be acute, and II.13 can be made to apply as pictured in al-Kūhī's diagram by reversing the roles of B and G at the outset of the problem.

Also, since $\angle DAG$ is given and $\angle GAD$ is right, ΔADG is known in form (*Data*, 40).[20] Thus $\frac{GD}{DA}$ is known.

Draw AE perpendicular to BG (def. E), and draw GZ so that $\frac{GD}{DA} = \frac{AE}{GZ}$.
Then GZ is known in magnitude, since AE is known in magnitude[21] (*Data*, 2).
Now $\frac{AE}{GZ} = \frac{GD}{DA} = \frac{GD \cdot AB}{DA \cdot AB} = \frac{AE \cdot BG}{DA \cdot AB}$; the latter is true because both $GD \cdot AB$ and $AE \cdot BG$ are twice ΔABG.
But $\frac{AE}{GZ} = \frac{AE \cdot BG}{GZ \cdot BG}$; therefore $DA \cdot AB = GZ \cdot BG$.

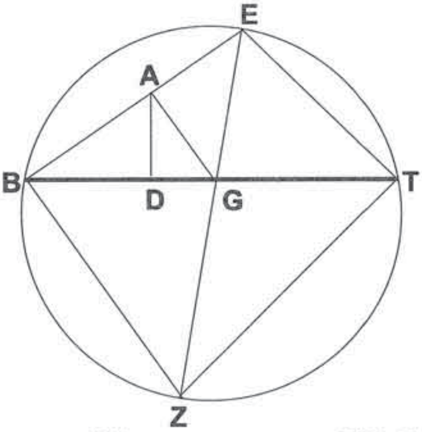
Hence $BG^2 + 2GZ \cdot BG$ is known (by equality with a known magnitude); hence $BG^2 + 2GZ \cdot BG + GZ^2$ is known (since GZ is known, its square is known by *Data*, 52, and the sum is known by *Data*, 3), and this latter quantity is equal to BZ^2.

Hence BZ is known in magnitude (*Data*, 55); hence $BG (= BZ - GZ)$ is known in magnitude (*Data*, 4). This reduces the problem to Problem 4.

[20] *Data*, 40 requires that all three angles of the triangle are known, but of course if two angles are known then so is the third. Al-Kūhī often omits this trivial step in this work, and we omit it as well from now on.

[21] It can be shown that AE is known in magnitude by the same reasoning used in Problem 2 to show that AD is known in magnitude.

Problem 6: Draw two line segments from A to the line BG, containing a given angle, so that $AB+AG$ is equal to a given length.



Analysis:
Draw AD perpendicular to BG. Then $\dfrac{AB+AG}{AD}$ is known (*Data*, 1).[22]

Produce BA to E so that $AE=AG$; then $\dfrac{EB}{AD}$ is known; thus $\dfrac{EB\cdot BG}{AD\cdot BG}$ is known (*Data*, 70).

Now $\dfrac{AD\cdot BG}{\Delta ABG}\left(=\dfrac{2}{1}\right)$ is known and $\dfrac{\Delta ABG}{GA\cdot AB}$ is known (*Data*, 66, since $\angle BAG$ is known).

Therefore $\dfrac{EB\cdot BG}{GA\cdot AB}\left(=\dfrac{EB\cdot BG}{EA\cdot AB}\right)$ is known (*Data*, 8, applied to the three area ratios above).

Draw BZ parallel to AG; join EG and produce it to meet BZ (def. Z). Produce BG to T so that $EG\cdot GZ=BG\cdot GT$. Then ΔEBZ is isosceles, since it is similar to ΔEAG.

Now $\dfrac{EA^2}{EG^2}$ is known (since $\angle EAG$ is 180° minus the given angle and the other angles are thus known because ΔEAG is isosceles; then by

[22] This is true because the perpendicular AD can be known by the same argument as in Props. 2 and 5.

Data, 40 ΔEAG is known in form; thus $\frac{EA}{EG}$ is known; thus $\frac{EA^2}{EG^2}$ is known by *Data*, 70), and it is equal to $\frac{EA \cdot AB}{EG \cdot GZ}$ (since $\frac{EA}{EG} = \frac{EB}{EZ} = \frac{AB}{GZ}$). Thus $\frac{EB \cdot BG}{EG \cdot GZ}$ is known (*Data*, 8).

But $\frac{EB \cdot BG}{EG \cdot GZ} = \frac{EB \cdot BG}{BG \cdot GT} = \frac{EB}{GT}$, and EB is known, so GT is known (*Data*, 2).

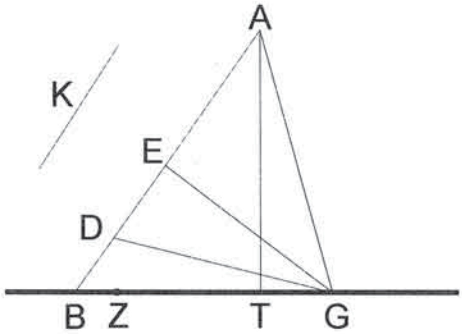
Now, since $EG \cdot GZ = BG \cdot GT$, we may draw a circle through E, B, Z, and T (by the converse to *Elements* III, 35). Also draw ET and TZ. Then, since $EB = BZ$, $\angle ETB = \angle ZTB$ (*Elements* III, 28 and 27[23]); also $\angle ZTB = \angle ZEB$ (*Elements* III, 21); therefore $\angle ETB = \angle ZEB$. Hence ΔEBT is similar to ΔEBG, since they share $\angle EBG$.

Thus $\frac{TB}{BE} = \frac{EB}{BG}$; thus $TB \cdot BG = EB^2$. But EB is known; thus $TB \cdot BG$ is known (*Data*, 52). And TG is known; thus BG is known (*Data*, 84). Thus $\frac{BG}{BA + AG}$ is known (*Data*, 1).

But $\angle BAG$ is given, so ΔBAG is known in form (*Data*, 45). Hence $\frac{BA}{AG}$ is known, which reduces the problem to Problem 1.

23 *Elements* III, 21 requires that the chords EB and BZ be in the same position, but this is of course not necessary. The result needed here may be obtained by applying III, 28 to find that the circumferences EB and BZ are equal, then applying III, 27 to find that the angles are equal.

Problem 7: Draw two line segments from A to the line BG, containing a given angle, so that $AB - AG$ is equal to a given length.



Analysis:

Define D on AB so that $BD = AB - AG$; then $DA = AG$. Draw GE perpendicular to AB.

$$
\begin{aligned}
\text{Then } BG^2 + 2BA \cdot AE &= BA^2 + AG^2 \text{ (\textit{Elements} II,13)}^{24} \\
&= BA^2 + AD^2 \\
&= BD^2 + 2BA \cdot AD \text{ (\textit{Elements} II,7)} \\
&= BD^2 + 2BA \cdot AE + 2BA \cdot ED,
\end{aligned}
$$

so, by cancellation, $BG^2 = BD^2 + 2BA \cdot ED$.

Define Z[25] on BG so that $GB \cdot BZ = BD^2$.

Then $BG \cdot BZ + 2BA \cdot ED = BG^2 = BG \cdot BZ + BG \cdot GZ$ (the former by substitution, the latter since $BG = BZ + GZ$), hence $2BA \cdot ED = BG \cdot GZ$, hence $BA \cdot ED = BG \cdot \left(\frac{1}{2}GZ\right)$.

Now $\angle GED$ is right, and $\angle GDE$ is known;[26] thus ΔGED is known in form (*Data*, 40) and so $\dfrac{GE}{ED}$ is known.

[24] As in Problem 5, this is true only if the given angle BAG is acute.

[25] The letter Z is missing from the diagram for problem 7 in the manuscript. That GB extends beyond B in the diagram suggests that the source for BN2457 might have had Z on this extension, but the proof clearly demands Z be chosen between B and G.

[26] This is presumably because triangle GAD is isosceles and angle GAD is given.

Draw a perpendicular from A to BG (def. T). Define line K so that $\frac{AT}{K} = \frac{GE}{ED}$. Since AT is known,[27] K is known as well (*Data*, 2).

Now $\frac{GE}{ED} = \frac{AB \cdot GE}{AB \cdot ED}$ and $\frac{AT}{K} = \frac{BG \cdot AT}{BG \cdot K}$; hence

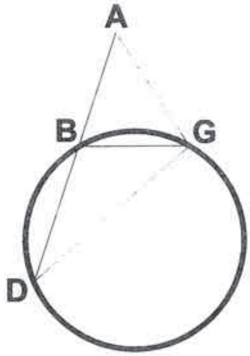
$\frac{AB \cdot GE}{AB \cdot ED} = \frac{BG \cdot AT}{BG \cdot K}$. But $AB \cdot GE = BG \cdot AT$, since they are both twice ΔABG; thus $AB \cdot ED = BG \cdot K$. And $AB \cdot ED = BG \cdot \left(\frac{1}{2} GZ\right)$; thus $K = \frac{1}{2} GZ$. Since K is known, GZ is known also (*Data*, 2).

Now $BD^2 = GB \cdot BZ$ (hence the latter is known), and both $BD(= AB - AG)$ and BZ are known (the latter by *Data*, 84); so BG is known (*Data*, 57).

Therefore $BG \cdot AT = 2\Delta ABG$ is known (by *Data*, 1 $BG \cdot AT$ is known in form; then by *Data*, 52 it is known in magnitude); hence ΔABG is known in magnitude (*Data*, 2). This reduces the problem to Problem 3.

For Problems 8-11, BG is a circle rather than a line. Each problem contains two diagrams, with A outside and inside the circle respectively. For the latter, A appears to be placed at the centre of the circle, but such a placement would render all four problems trivial. We assume an arbitrary interior point is meant. The analyses work equally well for both cases.

27 As in previous problems, this is not a direct consequence of a single *Data* proposition, but is easily shown.



Problem 8: Draw two line segments from A to the circle BG, containing a given angle, so that $\frac{AB}{AG}$ is equal to a given ratio.

Analysis:

Join BG; then ΔABG is known in form (by *Data*, 41, since $\frac{AB}{AG}$ and $\angle BAG$ are known), and so $\angle ABG$ is known.

Produce AB to D and join DG. Then $\angle GBD(=180° - \angle ABG)$ is known; hence DG is known in magnitude (*Data*, 87); hence DG^2 is known (*Data*, 52).

Now $\frac{AB}{AG} = \frac{AB \cdot AD}{AG \cdot AD}$; but $AB \cdot AD$ is known (*Data*, 91), so $AG \cdot AD$ is known (*Data*, 2).

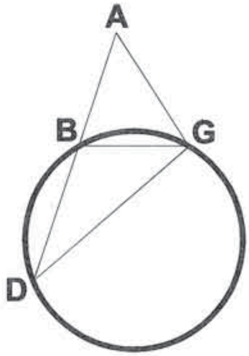
Hence $\frac{AG \cdot AD}{DG^2}$ is known (*Data*, 1), and since $\angle GAD$ is known, ΔGAD is known in form (*Data*, 80).

Thus $\frac{AG}{DG}$ is known; but DG is known in magnitude, so AG is known in magnitude (*Data*, 2). Hence point G is known.[28] But $\angle GAB$ is known, so point B is also known.[29]

[28] This may be seen by drawing a circle with centre A and radius AG. The intersection point G of this circle with the given circle is known by *Data*, 25.

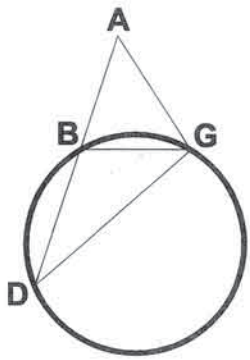
[29] By *Data*, 26 AG is known in position. Then, by *Data*, 29, AB is known in position, and by *Data*, 25, the intersection point B is known.

Problem 9: Draw two line segments from A to the circle BG, containing a given angle, so that $BA \cdot AG$ is equal to a given area.



Analysis:
[Produce AB to D and join BG and DG, as in Problem 8.]
$BA \cdot AD$ is known (*Data*, 91); thus $\frac{BA \cdot AG}{BA \cdot AD} = \frac{AG}{AD}$ is known.
And $\angle GAD$ is known, so by Problem 8, point G (as well as point D) is known.[30] Therefore point B is known.[31]

Problem 10: Draw two line segments from A to the circle BG, containing a given angle, so that ΔABG is equal to a given area.

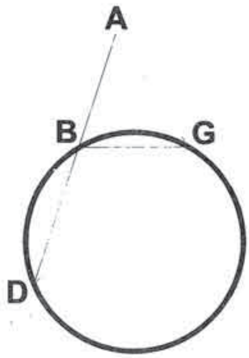


Analysis:
Since $\angle BAG$ is known, $\frac{\Delta ABG}{BA \cdot AG}$ is known (*Data*, 66). Therefore $BA \cdot AG$ is known (*Data*, 2), and the problem is reduced to Problem 9.

[30] This is an interesting step, since it is not a reduction in the sense that it is used in the rest of the treatise, but rather an appeal to the solution of a previous problem within the analysis. Note also that the solution of Problem 8 implicitly assumed that the line segments produced from A do not cut the circle prior to reaching the two desired points, but al-Kūhī uses Problem 8 in this manner here. The analysis in Problem 8 can be made to fit the situation here with only a trivial modification, namely, constructing point D as the intersection of AB with the circle rather than producing AB to D.

[31] This may be seen as in Problem 8.

Problem 11: Draw two line segments from A to the circle BG, containing a given angle, so that BG is equal to a given length.



Analysis:

[Produce AB to D and join BG and DG, as in Problem 8.]

Since BG is given, $\angle BDG$ is known (*Data*, 88). Since $\angle DAG$ is known as well, ΔDAG is known in form (*Data*, 40).

Therefore $\dfrac{DA}{AG} = \dfrac{DA \cdot AB}{AG \cdot AB}$ is known; but $DA \cdot AB$ is known (*Data*, 91), so $AG \cdot AB$ is known. This reduces the problem to Problem 9.

Al-Kūhī concludes the treatise by noting a few of the special cases that can emerge when BG is a circle:

- A lies on the circle. The solution is taken to be obvious and omitted.
- A lies at the centre of the circle. In this case the problem is either impossible or admits infinitely many solutions.
- A lies within the circle, but not at the centre. Al-Kūhī says that the number of cases that emerge would result in a work of Apollonian proportions, and states an intent to return to it some day.

Acknowledgements

We gratefully acknowledge the assistance of the following individuals: Prof. Fuat Sezgin, for making available to us a copy of the manuscript of the treatise; Mr. A. A. Hannawi, for his assistance in preparing a preliminary edition of the Arabic text and translation; and the referee, for making a number of helpful suggestions.

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بسم الله الرحمن الرحيم

كتاب إخراج الخطّين من نقطة على زاوية معلومة بطريق التحليل

استخرجه ويجن بن رستم المعروف بأبي[1] سهل القوهي

قال: غرضنا في هذا الكتاب إخراج خطّين مستقيمين من نقطة معلومة على زاوية معلومة إلى خطّ معلوم الوضع حتّى تكون نسبة أحدهما إلى الآخر، أو السطح الذي يحيطان به، أو قدر المثلّث الحادث أو قاعدته، أو مجموع مربّعيهما، أو مجموعهما، أو فضل ما بينهما معلوماً. فلتكن النقطة المعلومة نقطة آ، والخطّ المعلوم الوضع ب ج. وليكن أولاً مستقيماً.

ونريد أن نخرخ من نقطة آ خطّين مستقيمين إلى خطّ ب ج، ويحيطان بزاوية معلومة، حتّى تكون نسبة أحدهما إلى الآخر معلومة.

فعلى التحليل ننزل أنّ الخطّين المخرجين آ ب، آ ج، ونسبة آ ب إلى آ ج معلومة، وزاوية ب آ ج معلومة. فلأنّ نسبة خطّ ب آ إلى آ ج معلومة، وزاوية ب آ ج معلومة، فمثلّث آ ب ج معلوم الصورة. فزاوية آ ب ج معلومة. ونقطة آ معلومة، فخطّ آ ب معلوم الوضع، فخطّ آ ج معلوم الوضع، لأنّ زاوية ب آ ج معلومة. وخطّ ب ج معلوم الوضع، فكلّ واحدة[2] من نقطتي ب، ج معلومة. وذلك ما أردنا أن نعلم.

نريد أن نخرج من نقطة آ المعلومة خطّين مستقيمين إلى خطّ ب ج المستقيم المعلوم الوضع، ويحيطان بزاوية معلومة، حتّى يكون سطح أحدهما في الآخر معلوماً.

فعلى التحليل ننزل أنّ الخطّين المخرجين ب آ، آ ج، ويحيطان بزاوية ب آ ج المعلومة، وسطح ب آ في آ ج معلوم. لكن نسبة سطح ب آ في آ ج إلى مثلّث ب آ ج معلومة، لأنّ زاوية ب آ ج معلومة. فمثلّث ب آ ج معلوم القدر. ونسبة مثلّث آ ب ج إلى سطح ب ج في العمود الذي يخرج من نقطة آ إلى خطّ ب ج، وهو آ د، معلومة. فسطح ب ج في آ د معلوم القدر. وخطّ آ د معلوم القدر، فخطّ ب ج معلوم القدر، فمربّعه معلوم. فنسبة مربّع ب ج إلى سطح ب آ في آ ج معلومة. وزاوية ب آ ج معلومة، فمثلّث آ ب ج معلوم الصورة. فنسبة ب آ

[1]بأبي - ببا.

[2]واحدة - واحد.

إلى ا ج معلومة. وزاوية ب ا ج معلومة، فكل[3] واحدة من نقطتي ب، ج معلومة، كما بيّنّا في الشكل الذي تقدم. وذلك ما أردنا أن نعلم.

نريد أن نخرج من نقطة ا المعلومة خطّين مستقيمين إلى خط ب ج المستقيم المعلوم الوضع، ويحيطان بزاوية معلومة، حتّى يكون المثلّث الحادث معلوم القدر.

فعلى التحليل ننزل أن الخطّين المخرجين ا ب، ا ج، ويحيطان بزاوية ب ا ج المعلومة، ومثلّث ا ب ج معلوم القدر. لكن نسبة مثلّث ا ب ج إلى سطح ب ا في ا ج معلومة، لأن زاوية [٤٨ ب] ب ا ج معلومة. فسطح ب ا في ا ج معلوم القدر. وزاوية ب ا ج معلومة، فكل واحدة من نقطتي ب، ج معلومة، كما تبيّن في الشكل الذي تقدم. وذلك ما أردنا أن نعلم.

نريد أن نخرج من نقطة ا المعلومة خطّين مستقيمين إلى خط ب ج المستقيم المعلوم الوضع، ويحيطان بزاوية معلومة، حتّى تكون قاعدة مثلّث ا ب ج، وهي[4] ب ج، معلومة.

فعلى التحليل ننزل أن الخطّين المخرجين ا ب، ا ج، ويحيطان بزاوية ب ا ج المعلومة، وأن قاعدة ب ج معلومة، فسطح ب ج في العمود الذي من نقطة ا إلى قاعدة ب ج، وهو ا د، معلوم. لكن نسبة سطح ب ج في ا د إلى مثلّث ا ب ج معلومة، فمثلّث ا ب ج معلوم القدر. فنقطتا[5] ب، ج معلومتان[6]، كما تبيّن في الشكل المتقدم. وذلك ما أردنا أن نعلم.

نريد أن نخرج من نقطة ا المعلومة خطّين مستقيمين إلى خط ب ج المستقيم المعلوم الوضع، ويحيطان بزاوية معلومة، حتّى يكون مجموع مربّعيهما مساوياً لسطح معلوم. فعلى التحليل ننزل أن الخطّين المخرجين ب ا، ا ج، ويحيطان بزاوية ب ا ج المعلومة، ومربّعا خطّي ب ا، ا ج مجموعان معلوم، وأن خط ج د عمود على خط ا ب. فلأن <مجموع> مربّعي خطّي ب ا، ا ج مساو لمربّع خط ب ج مع ضعف سطح ب ا في ا د، فمربّع خط ب ج مع ضعف سطح ب ا في ا د معلوم.

[3] فكل - وكل.

[4] وهي - وهو.

[5] فنقطتا - فنقطتي.

[6] معلومتان - معلومتين.

وأيضاً، لأن زاوية د ا ج معلومة وزاوية ج د ا قائمة، فمثلث ا د ج معلوم الصورة. فنسبة خط ج د إلى د ا معلومة. فإن نحن جعلنا هذه النسبة كنسبة ا ه العمود من نقطة ا على خط ب ج إلى خط آخر، وليكن ج ز، كان خط ج ز معلوم القدر، لأن خط ا ه معلوم القدر.

ولأن نسبة خط ا ه إلى خط ج ز كنسبة خط ج د إلى خط د ا، ونسبة خط ج د إلى د ا كنسبة سطح ج د في ا ب إلى سطح د ا في ا ب، لأن ا ب ارتفاع مشترك لهما، وسطح ج د في ا ب مساوٍ لسطح ا ه في ب ج، لأن كل واحد منها مساوٍ لضعف مثلث ا ب ج، فنسبة سطح ا ه في ب ج إلى سطح د ا في ا ب كنسبة خط ا ه إلى ج ز. ولكن نسبة خط ا ه إلى ج ز كنسبة سطح ا ه في ب ج إلى سطح ج ز في ج ب، لأن ب ج ارتفاع مشترك لهما، فنسبة سطح ا ه في ب ج إلى كل واحد من سطحي ب ج في ج ز وب ا في ا د واحدة. فسطح ب ا في ا د مساوٍ لسطح ب ج في ج ز. فضعف سطح ب ا في ا د مساوٍ لضعف سطح ب ج في ج ز.

فمربّع خط ب ج مع ضعف سطح ب ج في ج ز معلوم. ونزيد عليه مربّع خط ج ز المعلوم، فيكون جملة ذلك مربّع ب ز، فمربّع خط ب ز معلوم. فخط ب ز معلوم القدر، وخط ج ز معلوم القدر. فخط ب ج الباقي معلوم القدر. وزاوية ب ا ج معلومة. فكل واحدة من نقطتي ب، ج معلومة، على ما تبيّن في الشكل المتقدم. وذلك ما أردنا أن نعلم.

[٤٩ أ] نريد أن نخرج من نقطة ا المعلومة خطّين مستقيمين إلى خط ب ج المستقيم المعلوم الوضع، ويحيطان بزاوية معلومة، حتّى يكون مجموعهما معلوم القدر.

فعلى التحليل ننزل أن الخطّين المخرجين ب ا، ا ج، ويحيطان بزاوية ب ا ج المعلومة، ومجموع خطّي ب ا، ا ج معلوم القدر. ونسبة خطّي ب ا، ا ج إلى العمود الذي يخرج من نقطة ا إلى خط ب ج، وهو ا د، معلومة. فإن نحن توهّمنا خط ا ه مساوياً لخط ا ج، وخط ب ا ه مستقيم، تكون نسبة خط ه ب إلى خط ا د معلومة. ونجعل خط ب ج ارتفاعاً مشتركاً لهما، فنسبة سطح ه ب في ب ج إلى سطح ا د في ب ج معلومة. ونسبة سطح ا د في ب ج إلى مثلّث ا ب ج معلومة لأنّه ضعفه. ونسبة مثلّث ا ب ج إلى سطح ج ا في ا ب معلومة، لأن زاوية ب ا ج معلومة، فنسبة سطح ه ب في ب ج إلى سطح ج ا في ا ب، أعني إلى سطح ه ا في ا ب، معلومة.

فإن نحن توهّمنا خط ب ز موازياً لخط ا ج، وخط ه ج ز مستقيماً، وسطح

ه ج في ج ز مساوياً لسطح ب ج في ج ط، كان مثلث ه ب ز متساوي الساقين، لأنه شبيه بمثلث ه ا ج. ومثلث ه ا ج معلوم الصورة، لأن زاوية ه ا ج معلومة. فنسبة سطح ه ا في ا ب إلى سطح ه ج في ج ز معلومة، لأنها كنسبة مربع ه ا إلى مربع ه ج. فنسبة سطح ه ب في ب ج إلى سطح ه ج في ج ز معلومة. وسطح ه ج في ج ز مساوٍ لسطح ب ج في ج ط. فنسبة سطح ه ب في ب ج إلى سطح ب ج في ج ط معلومة. وهي كنسبة ه ب المعلوم إلى خط ج ط، لأن ب ج هو ارتفاع مشترك لهما، فخط ج ط معلوم.

وأيضاً، فلأن سطح ه ج في ج ز مساوٍ لسطح ب ج في ج ط، فقد نختط على نقطة ه، ب، ز، ط دائرة ما، ونصل خطي ه ط، ط ز. فزاوية ه ط ب مساوية لزاوية ز ط ب، لأن وتر ه ب مساوٍ لوتر ب ز. وزاوية ب ط ز مساوية لزاوية ز ه ب، لأن قاعدتهما قوس واحدة، فزاوية ب ط ه مساوية لزاوية ج ه ب. وزاوية ه ب ج مشتركة، فمثلث ه ب ط يشبه مثلث ه ب ج. فنسبة خط ط ب إلى ب ه كنسبة خط ه ب إلى ب ج، فسطح ط ب في ب ج مساوٍ لمربع خط ه ب. ومربع خط ه ب معلوم، لأن ه ب معلوم، فسطح ط ب في ب ج معلوم. وخط ط ج قد تبين أنه معلوم، فخط ب ج معلوم. فنسبته إلى خطي ب ا، ا ج مجموعين معلومة. وزاوية ب ا ج معلومة، فمثلث ا ب ج معلوم الصورة. فنسبة ب ا إلى ا ج معلومة، فكل واحدة من نقطتي ب، ج معلومة. وذلك ما أردنا أن نعلم.

نريد أن نخرج من نقطة ا المعلومة خطين مستقيمين إلى خط ب ج المستقيم المعلوم الوضع، ويحيطان بزاوية معلومة، حتى يكون فضل ما بينهما[7] معلوماً.

فعلى التحليل ننزل أن الخطين المخرجين ب ا، ا ج، ويحيطان بزاوية ب ا ج المعلومة، وفضل خط ب ا على ا ج معلوم، وهو ب د، فخط د ا الباقي مساوٍ لخط ا ج. فإن نحن توهمنا خط ج ه عموداً على ا ب، فمربع خط ب ج يكون، مع ضعف سطح ب ا في ا ه، مساوياً لمربعي خطي ب ا، ا ج. لكن <مجموع> مربعي خطي ب ا، ا ج مساوٍ لمربعي خطي ب ا، ا د، لأن ا د [٤٩ ب] مساوٍ لخط ا ج. فمربع خط ب ج مع ضعف سطح ب ا في ا ه مساوٍ لمربعي خطي ب ا، ا د. ومربعا خطي ب ا، ا د مساوٍ لضعف سطح ب ا في ا د مع مربع خط ب د، فمربع خط ب ج مع ضعف سطح ب ا في ا ه مساوٍ

[7] بينهما - بينها.

لمربّع خط ب د مع ضعف سطح ب ا في ا د. لكن ضعف سطح ب ا في ا د مساوٍ لضعف سطح ب ا في ا ه مع ضعف سطح ب ا في ه د، فمربّع خط ب ج، مع ضعف سطح ب ا في ا ه، مساوٍ لمربّع خط ب د مع ضعف سطح ب ا في ا ه، وضعف سطح ب ا في د ه. فإذا ألقي ضعف سطح ب ا في ا ه المشترك يبقى مربّع ب ج مساوياً لمربّع ب د مع ضعف سطح ب ا في ه د.

فإن نحن توهّمنا سطح ج ب في ب ز مساوياً لمربّع ب د كان سطح ج ب في ب ز مع ضعف سطح ب ا في د ه مساوياً لمربّع خط ب ج. لكن مربّع خط ب ج مساوٍ لسطحين، أحدهما ج ب في ب ز والآخر ب ج في ج ز. فإذا ألقي سطح ج ب في ب ز المشترك بقي سطح ب ج في ج ز مساوياً لضعف سطح ب ا في د ه، فسطح ب ا في د ه مساوٍ لسطح ب ج في نصف خط ج ز. وأيضاً، لأن زاوية د ه ج قائمة، وزاوية ج د ه معلومة، فمثلّث ج ه د معلوم الصورة، فنسبة خط ج ه إلى ه د معلومة. فإن نحن جعلنا نسبة خط ا ط العمود من نقطة ا على خط ب ج إلى خط آخر، وهو ك، كنسبة خط ج ه إلى خط ه د، كان خط ك معلوماً، لأن خط ا ط معلوم، ولأن نسبة خط ا ط إلى خط ك كنسبة خط ج ه إلى خط ه د. ونسبة خط ج ه إلى خط ه د كنسبة سطح ا ب في ج ه إلى ا ب في ه د، ونسبة خط ا ط إلى ك كنسبة سطح ب ج في ا ط إلى سطح ب ج في ك، فنسبة سطح ب ا في ج ه إلى سطح ب ا في د ه كنسبة سطح[8] ب ج في ا ط إلى سطح ب ج في ك. وسطح ب ا في ج ه مساوٍ لسطح ب ج في ا ط، لأن كل واحد منهما مساوٍ لضعف مثلّث ا ب ج. فسطح ب ا في ه د مساوٍ لسطح ب ج في ك. وقد تبيّن أن سطح ب ا في ه د مساوٍ لسطح ب ج في نصف خط ز ج[9]، فسطح ب ج في نصف خط ز ج مساوٍ لسطح ب ج في ك. فخط ك مساوٍ لنصف خط ز ج[10]. فخط ز ج معلوم لأن ك، الذي هو نصفه، معلوم. وسطح ج ب في ب ز معلوم لأنّه مساوٍ لمربّع خط ب د المعلوم، فخط ب ز معلوم، فجميع ب ج معلوم. وخط ا ط معلوم، فسطح أحدهما في الآخر معلوم، وهو ضعف مثلّث ا ب ج، فمثلّث ا ب ج معلوم. وزاوية ب ا ج معلومة، وكل واحدة من نقطتي ب، ج معلومة، كما تبيّن في الشكل الذي تقدم. وذلك ما

[8] سطح - مكتوبة في الهامش.

[9] ز ج - د ج.

[10] ز ج - د ج.

أردنا أن نبيّن.

وليكن الخطّ المعلوم الوضع، وهو ب ج، من محيط دائرة معلومة الوضع، ونريد أن نخرج من نقطة آ المعلومة خطّين مستقيمين إلى خط ب ج، الذي هو محيط الدائرة، ويحيطان بزاوية معلومة، حتّى [٥٠ أ] تكون نسبة أحدهما إلى الآخر معلومة.

فعلى التحليل ننزل أنّ الخطّين المخرجين آ ب، ب ج، وزاوية ب آ ج معلومة، ونسبة ب آ إلى آ ج معلومة. ونصل خط ب ج، فمثلّث آ ب ج المستقيم الأضلاع معلوم الصورة، فزاوية ب معلومة. فإن نحن توهّمنا أن خط ب آ د مستقيم كان وتر قوس د ج، وهو خط د ج، معلوماً، لأنّ زاوية ب على محيط دائرة معلومة الوضع معلومة. فمربّع خط د ج معلوم. وأيضاً، فلأن نسبة خط ب آ إلى آ ج معلومة، فإنه إذا جُعل آ د ارتفاعاً مشتركاً لهما كانت نسبة سطح ب آ في آ د إلى سطح ج آ في آ د معلومة. لكن سطح ب آ في آ د معلوم، لأنّ نقطة آ معلومة، فسطح ج آ في آ د معلوم. فنسبة سطح ج آ في آ د إلى مربّع ج د معلومة. وزاوية ج آ د معلومة، فمثلّث ج آ د معلوم الصورة. فنسبة خط آ ج إلى خط ج د معلومة. وخط ج د معلوم، فخط آ ج معلوم. ونقطة آ معلومة. فخط آ ج معلوم الوضع. ومحيط الدائرة معلوم الوضع، فنقطة ج معلومة. ولأنّ زاوية ج آ ب معلومة، فنقطة ب أيضاً معلومة. فكلّ واحدة من نقطتي ج، ب معلومة، وذلك ما أردنا أن نعلم.

نريد أن نخرج من نقطة آ المعلومة خطّين مستقيمين إلى خط ب ج الذي هو من محيط دائرة معلومة الوضع، ويحيطان بزاوية معلومة، حتّى يكون سطح أحدهما في الآخر معلوماً.

فعلى التحليل ننزل أنّ الخطّين المخرجين ب آ، آ ج، ويحيطان بزاوية ب آ ج المعلومة، وسطح ب آ في آ ج معلوم. وسطح ب آ في آ د معلوم، لأنّ نقطة آ معلومة، فنسبة سطح ب آ في آ ج إلى سطح ب آ في آ د معلومة. وهي كنسبة خط ج آ إلى آ د، لأنّ خط ب آ ارتفاع مشترك لهما. فنسبة خط ج آ إلى آ د معلومة. وزاوية ج آ د معلومة، فنقطة ج معلومة، كما تبيّن في الشكل الذي تقدم. فنقطة ب أيضاً معلومة، لأنّ زاوية ج آ ب معلومة. وذلك ما أردنا أن نعلم.

نريد أن نخرج من نقطة آ المعلومة خطّين مستقيمين إلى خط ب ج المعلوم الوضع، الذي من محيط دائرة، ويحيطان بزاوية معلومة، حتّى يكون مثلّث ا ب ج المستقيم الأضلاع مساوياً لسطح معلوم. فعلى التحليل ننزل أن الخطّين

المخرجين ب ا، ا ج، ويحيطان بزاوية ب ا ج المعلومة. ومثلث ب ا ج معلوم القدر. لكن نسبة مثلث ا ب ج إلى سطح ب ا في ا ج معلومة، لأن زاوية ب ا ج معلومة. فسطح ب ا في ا ج معلوم القدر. وزاوية ب ا ج معلومة، فكلٍ واحدة من نقطتي [٥٠ ب] ب، ج معلومة، على ما تبيّن في الشكل الذي تقدم. وذلك ما أردنا أن نعلم.

نريد أن نخرج من نقطة ا المعلومة خطّين مستقيمين إلى خط ب ج الذي هو من محيط دائرة معلومة الوضع، ويحيطان بزاوية معلومة، حتّى تكون قاعدة المثلّث، وهي[11] ب ج، معلومة.

فعلى التحليل ننزل أن الخطّين المخرجين ب ا، ا ج، ويحيطان بزاوية ب ا ج المعلومة، وأن ب ج معلوم. ونخرج خط ب ا على استقامة إلى د، ونصل د ج. فلأن ب ج معلوم، فزاوية ب د ج معلومة. وزاوية د ا ج معلومة، فمثلّث د ا ج معلوم الصورة. فنسبة د ا إلى ا ج معلومة، وهي كنسبة سطح د ا في ا ب[12] إلى سطح ج ا في ا ب. لكن سطح د ا[13] في ا ب معلوم، فسطح ج ا في ا ب معلوم. وزاويةٍ ب ا ج معلومة، فنقطتا[14] ب، ج معلومتان[15]، كما تبيّن في الشكل المتقدم. وذلك ما أردنا أن نعلم.

فإما إذا كانت النقطة المعلومة على محيط الدائرة كان آستنباط النقطتين اللتين ينتهي إليهما الخطّان[16] المخرجان أسهل وأقرب. ولفرط ظهوره ما تركنا ذكره. وأما إذا كانت النقطة المعلومة مركز الدائرة، فالمسائل إما أن تكون صحيحة وهي سيّالة، وإما أن تكون مستحيلة. وإن لم تكن النقطة المعلومة مركز الدائرة كانت النقط التي تستنبط أحوال مختلفة، وإنما تركت ذكرها لأنّها من معاني التحديد. ولو استعملنا فيه التقسيم والتحديد والتركيب ووقوع النقط بطريق أبلونيوس في بعض كتبه لخرج كتاباً كبيراً، ونحن

[11] وهي - وهو.

[12] ا ب - ا م.

[13] د ا - ك ا.

[14] فنقطتا - فنقطتي.

[15] معلومتان - معلومتين.

[16] الخطّان - الخطين.

نرجو[17] الفراغ لذلك فنعمله إن شاء الله.الحمد لله وصلّى الله على محمد وآله وسلّم.
عورض بالأصل.

[17]نرجو - نرجوا.