Abū Sahl al-Kūhī's "On the Ratio of the Segments of a Single Line that Falls on Three Lines"

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Introduction

Abū Sahl al-Kūhī, one of the leading mathematicians of the tenth century AD, made significant contributions to several fields, including centres of gravity and astronomical topics. However, his primary legacy is to geometry, in the tradition of the ancient Greeks. Many of his works are devoted to explaining and expanding on the writings of Euclid, Archimedes and Apollonius, including such diverse topics as the construction of the regular heptagon, the volume of a segment of a paraboloid, and a study of the complete compass, an instrument used for drawing conic sections. Even his treatise on the construction of the astrolabe would have been of interest primarily to geometers.

The subject of the present study, "On the Ratio of the Segments of a Single Line that Falls on Three Lines",[1] is one of several of al-Kūhī's works that are dedicated to the famous Būyid ruler and patron of the sciences, Sharaf al-Dawla. That the treatise is dedicated to Sharaf al-Dawla, with his title Zayn al-Milla, and that he reigned from 372/982 to 379/989, give an approximate date for the treatise. (Sharaf's father, ʿAḍud al-Dawla, began the family practice of taking honorific titles and, although we can find no definite information in the case of Sharaf al-Dawla, we

[1] This is the seventh treatise by al-Kūhī listed by F. Sezgin in [10, 319].

suspect that he took the honorific title virtually simultaneously with his accession to the throne.) In this self-contained treatise, al-Kūhī discusses varieties of a construction problem that could have occurred to the ancient Greek geometers.[2] The general problem of this treatise is to construct a straight line through a given point and cutting three given straight lines so that two of the segments of the constructed line, cut by the given lines, are to each other in a given ratio. This problem is reminiscent of Apollonius's *Cutting Off of a Ratio*, in which Apollonius solves varieties of the problem of constructing a line through a given point and cutting two lines, each containing a given point, so that the two segments cut off on the given lines, from the points given on them to the intersections with the constructed line, are in a given ratio.[3] This work is extant in an Arabic translation, and may well have been available to al-Kūhī.[4] Indeed, al-Kūhī's work follows Apollonius's style of dealing with the variety of possible relative positions of the given lines, and (like Apollonius) uses the methods of analysis and synthesis in order to arrive at solutions.

The role of analysis in the practice of geometry seems to have been an issue in the tenth century. Ibrāhīm ibn Sinān (AD 909-946) wrote a work on analysis and synthesis, in which he demonstrated that the geometers' use of analysis at the time contained, in essence, the synthesis.[5] By expanding the analysis to make explicit the relations already implicit in the occurrences of the word "known", the construction of the sought object can be determined. This leads to an analysis and synthesis that are so parallel to each other as to cover virtually the same ground, although in

[2] Other works of this sort by al-Kūhī include "On Tangent Circles" (a translation and commentary appear in [1]) and "On Drawing Two Lines from a Point at a Known Angle".

[3] See [9, 510-512] for a discussion of this work and [9, 606-619] for a translation of some selections.

[4] Ibrāhīm ibn Sinān, for instance, refers to it in his treatise on analysis and synthesis (see [6, 191]); al-Sijzī refers to it in solving a problem set by al-Kūhī [6, 217].

[5] A discussion of Ibrāhīm ibn Sinān's work on analysis and synthesis may be found in [2]. We shall argue elsewhere that Ibn Sinān was not advocating that geometers should expand analyses in practice.

different logical contexts. However, Ibn Sinān does not advocate expanding analyses in this way in practice.

The present work has particular relevance to the role of analysis in medieval Islamic geometry, since it contains both the analysis and the synthesis for all five problems, something not common in al-Kūhī's works. The analyses are carried through to a high degree of rigour, and every step may be justified in terms of one or two appeals to propositions in Euclid's *Data*. Also, the analyses and syntheses are parallel in structure and the diagrams for the analyses are almost all identical to those for the syntheses.

Since al-Kūhī's style of analysis may not be familiar to the reader, we outline it briefly here. One begins by assuming that the construction has been completed, but that only the elements provided in the problem statement are "known", i.e., completely determined in their position, magnitude or ratio (or to within a finite number of possibilities).[6] Starting 2from these givens, one uses the propositions of Euclid's *Data* or other treatises of analysis (in this work the *Data* is sufficient) to determine the positions, magnitudes or ratios of further geometric elements in the diagram, which are then said to be "known", until the object which is the goal of the construction is "known". At this point the analysis is complete.

Logically, this completed analysis does not constitute a construction; all one knows is that the sought-for element is a single well-defined object (or one of a finite number of them). Usually, however, analysis provides enough of a framework to build a synthesis, the more familiar deductive proof, fairly easily.

Of additional interest in this treatise is the use of a hyperbola to solve the final problem, in which al-Kūhī generalizes the problem by allowing one of the three given lines to be a curve. This proposition illustrates how al-Kūhī applies the notion of "known" both to conic sections and to arbitrary curves. In addition, we note (contrary to views expressed in [1, 267]) that al-Kūhī retains the Greek distinction between plane loci (straight lines and circles) and solid loci (usually, conic sections), although he does not use these terms, and follows the Greeks in preferring plane loci within a

[6] Al-Kūhī uses the word "known" in the same way that Euclid uses "given" in the *Data*. See [5], especially pp. 207-210, for a discussion of Arabic words used for this concept.

solution, even when the use of solid loci would produce a better or more general solution.

Translation

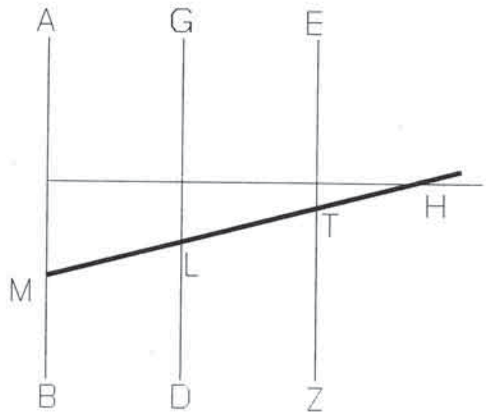
Our edition of the text is based on the unique known copy of this work in the Istanbul (Süleymaniye Library) manuscript AS4830, 165a - 168b. In our translation, parentheses are used as punctuation which (like other punctuation — periods, commas, etc.) is entirely lacking in the Arabic original. Square brackets enclose explanatory remarks that we have inserted in the text. Pointed brackets (used twice) enclose our restoration of the text. Our system for transliterating letters referring to points in the diagrams is that of Kennedy and Hermelink [7]. In our edition of the text we have used superscript numerals to refer to our critical notes. in the opening lines of the translation we have chosen to let the copyist's spelling of al-Kūhī's name (al-Qūhī) stand. In editing the text we have not noted the few places in which we have changed the orthography to standard classical orthography, nor the several places where we have corrected what seemed to be trivial scribal errors, such as misreadings of yā for tā (and conversely). Our policy on noting what could be construed as scribal errors in copying letters referring to points in geometrical diagrams is that if the form of the letter in the text could be construed as representing the correct letter then we took it as being correct. Otherwise we made a note. Not infrequently words near the ends of lines or the bottom of pages are obscured in the copies we are using from the microfilm, which was generously placed at our disposal by Prof. Fuat Sezgin. In many cases the word is sufficiently legible to warrant no note; in a few cases a note is necessary, and in one case the words were so obscure that we have placed our reading of the text in pointed brackets, as being essentially a restoration.

In the name of God, the merciful, the compassionate.
My trust is in God. Praise Him.

Treatise of the excellent shaykh,
Abū Sahl Wayjan ibn Rustam al-Qūhī

On the Ratio of the Segments of a Single Line that Falls on Three Lines

One of the abstruse theorems, which was not revealed in the time of any of the [earlier] kings but has been revealed in the time of our master, the king, the great lord, the *shāhānshāh*, Sharaf al-Dawla and Zayn al-Milla (may God prolong his life and extend his power), was that of drawing a straight line from a known point to three lines known in position so that the ratio of one of the two segments which are formed between two of the three lines to the other segment is known.



[Problem 1][7]

So, let the three lines be AB, GD and EZ. Then if these lines are parallel

[7] The heavy line and the corresponding letters in the diagram for this problem appear to have been added by Ibn Sirtāq, who also wrote the marginalia.

and the known ratio is as the ratio of the perpendicular which falls between EZ and GD to the perpendicular which falls between GD and AB then the problem is undetermined[8] since every [line] produced from every point yields this ratio, which varies not at all with the variation in the line or the point from place to place.

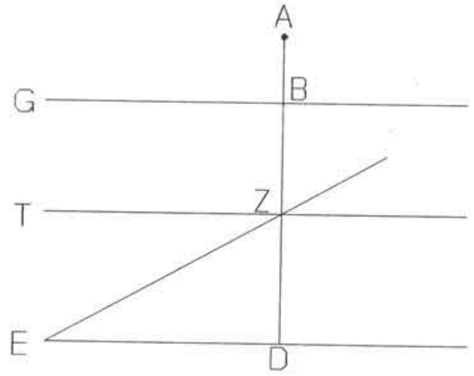
And if the known ratio is <not> as the ratio of the perpendicular which we mentioned to the other [perpendicular] it is impossible that there falls between these three lines, which are parallel, two lines in that ratio. This thing is clear, too clear to need further proof or explanation. And that is what we wanted to prove.

[Problem 2][9]

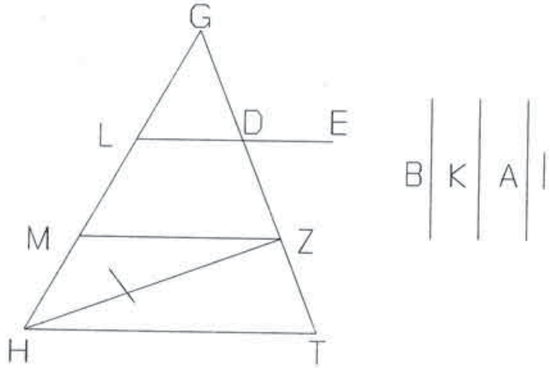
If only two of these three lines are parallel, then by way of analysis we suppose: The point A is known, only the two lines BG and DE are parallel and the two lines DE and ZE meet at the point E, the line sought for is ABDZ, and the ratio BD to DZ is known. Then the ratio of line AD to line DB is known since the two lines BG, DE are parallel. And so the ratio of line AD to line DZ is known. And so the point Z is on a straight line known in position parallel to line DE, which is also known in position, and it is also on the line ZE, which is known in position. So the point Z is known since it is on the common section of the two lines ZE, ZT known in position. And the point A is known, so the line ABDZ is known in position. Therefore each of the points A, B, D and Z is known, and that is what we wanted to prove.

[8] The Arabic word that we translate "undetermined" is "*sayyāl* (fluid)," which is also used in the medieval Arabic algebraic literature for indeterminate equations. For another appearance of the word in al-Kūhī's work, see J. Lennart Berggren, "The correspondence of Abū Sahl al-Kūhī and Abū Isḥāq al-Ṣābī: A translation with commentaries", *Journal for the History of Arabic Science* 7 (1983), 73.

[9] There are some problems with the text of the treatise in Problem 2, including the interchange of Z, D in the figure. In our translation, we have put them where they should be, and we justify, in the commentary, the choices we have made.



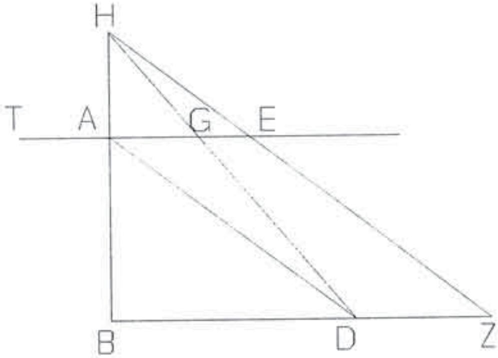
The synthesis: [Suppose that] the known ratio is the ratio of A to B, the known point is the point G, the three lines are DE, ZH and TH and that the two lines ZH and TH meet at point H. Draw the line GH. Then it will meet the line DE at the point [say] L, and we make the ratio of A to [some segment, say] K as the ratio of LH to GH. We [now choose a point M on GH so as to] make the ratio GH to MH as the ratio of K to B, and we draw through the point M a line parallel to the line HT. Then it will meet the line ZH, say at the point Z. Finally we draw the line GZ and we produce it to the point T. Then I say that the ratio of DT to ZT is as the ratio of A to B.



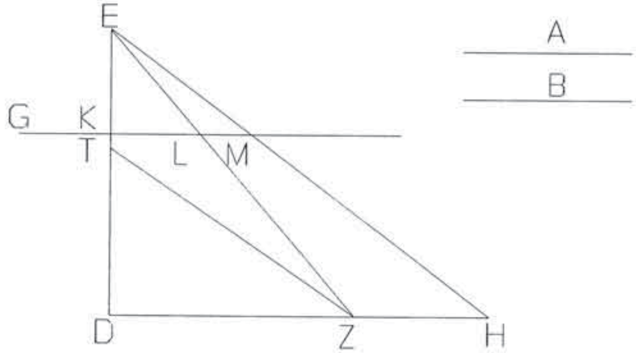
The proof of that is that the ratio DT to TG is as the ratio of LH to HG, since the two lines DL, TH are parallel, and the ratio of LH to HG is as the ratio of A to K. Therefore the ratio of DT to TG is as the ratio of A to K. Also the ratio of GT to ZT is as the ratio of GH to MH, which is as the ratio of K to B. Thus, *ex æquali*, the ratio of DT to ZT is as the ratio of A to B. And that is what we wanted to prove.

[Problem 3]

And if the three lines AB, GD and EZ are not parallel, but intersect at a single point, H, we want to draw from the known point T a straight line so that the ratio of its segment between the two lines HB and HD to its segment between HD and HZ is known.



The analysis: We suppose that the line sought for is TAGE, that the ratio of AG to GE is known, and that the line BDZ through the known point B is parallel to line AGE. Then the ratio BD to AG is as the ratio of DH to HG, and the ratio of DH to HG is as the ratio of ZD to EG. Therefore the ratio of DB to AG is as the ratio of ZD to EG. *Alternando* the ratio of DB to DZ is as the ratio of AG to GE, and the ratio of AG to GE is known. Thus, the ratio of BD to DZ is known. The point B also is known, so the point D is on the straight line known in position [and] parallel to HZ, and it is on the line HGD known in position. Therefore the point D is known since it is on the section common to two lines known in position. And the point B is known, so the line BDZ is known in position, so line AGE is known in position since it is parallel to line BDZ, which is known in position [and passes through the known point T]. Thus each of the points A, G and E is known, which is what we wanted to prove.

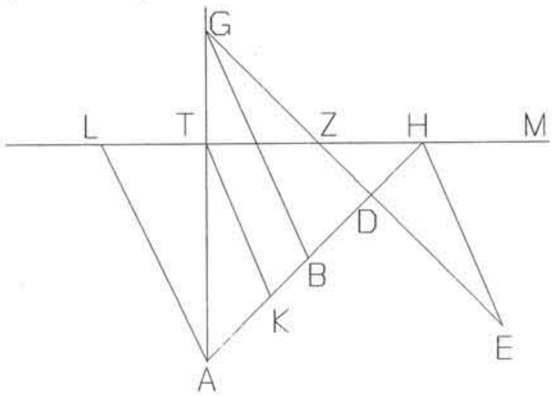


The synthesis:[10] The known ratio is the ratio of A to B, the known point is the point G, and the three known lines, which are not parallel and which intersect at a single point, are the lines DE, ZE and HE. We mark on the line DE an arbitrary point, D, and we [choose T on DE so as to] make the ratio of DT to TE as the ratio of A to B. We draw from the point T a line parallel to the line EH, so it will meet the line EZ, say at Z. We draw line ZDH and from the point G we draw the line GKLM parallel to line DZH. I say that the ratio of KL to LM is as the ratio of A to B.

Proof: The ratio of LK to ZD is as the ratio of LE to EZ, by the similarity of the two triangles, and the ratio of LE to EZ is as the ratio of ML to ZH. Therefore the ratio of LK to ZD is as the ratio of ML to HZ. *Alternando* the ratio of KL to LM is as the ratio of DZ to ZH, and the ratio of DZ to ZH is as the ratio of DT to TE, since the two lines TZ and EH are parallel. Therefore the ratio of KL to LM is as the ratio of DT to TE and the ratio of DT to TE is as the ratio of A to B. Therefore the ratio of KL to LM is as the ratio of A to B, which is what we wanted to prove.

[10] In the diagram for the synthesis in the manuscript the line TZ has been replaced by the line KZ, the letter G is missing, and in its place the letter T appears.

[Problem 4]

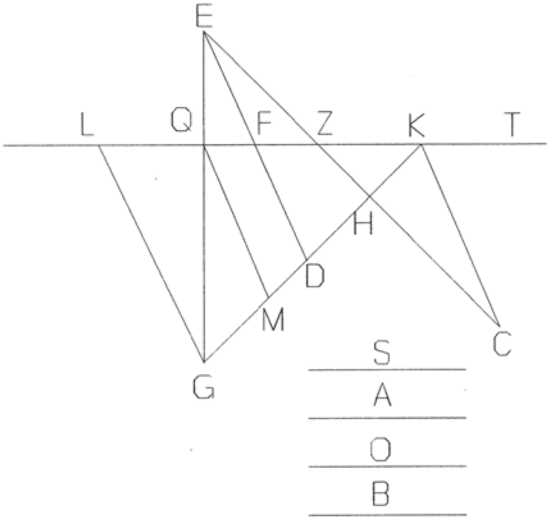


And if the three lines GB, DE and ZH are not parallel and do not intersect at only one point we want to draw from the known point A a line so that the ratio of its segment between the two lines BG and DE to its segment between the two lines DE and ZH is known.

The analysis: We suppose that the line sought is ABDH and that the ratio of BD to DH is known. Now [draw] the line HE parallel to the line BG. Then the ratio of BG to EH is known since it is equal to the known ratio of BD to DH. But the ratio of BG to EH is compounded of the ratio of BG to ZH and the ratio of ZH to HE. And the ratio of ZH to HE is known since each of the angles of triangle ZHE is known. Therefore, the ratio of BG to ZH is known.

And if we draw AG and make line TK parallel to line BG the ratio of TA to AG will be known. And the ratio of TA to AG is as the ratio of TK to GB, so the ratio of TK to GB is known. Moreover the ratio of GB to ZH was known, and so the ratio of TK to ZH is known. And if we make this known ratio equal to the ratio of the line AL, which is parallel to the known line BG, to the line ZM, the line ZM will be known. And the point Z is known, so the point M is known. Also, since the ratio of TK to ZH is as the ratio of AL to ZM, then *alternando* the ratio of TK to AL will be as the ratio of HZ to ZM. However, the ratio of TK to AL is as the ratio of TH to HL, and therefore the ratio of TH to HL is as the ratio of HZ to ZM. Therefore, *invertando*, the ratio of TL to LH is as the ratio of HM to

MZ. Therefore, the product of LT by ZM, which is known since each of the two [terms] is known, is equal to the product of LH by HM. Therefore the product of LH by HM is known, and the line LM is known, so the point H is known. And the point A is known, so line ABDH is known in position. Q.E.D.

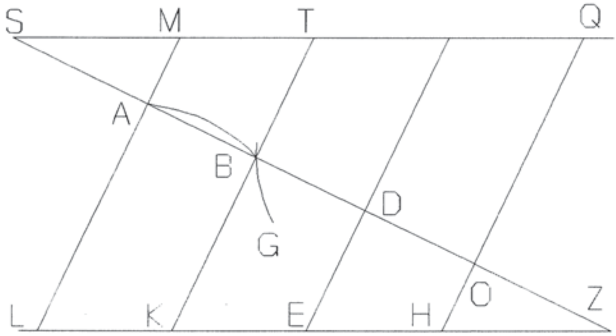


The synthesis: The known ratio is A to B and the known point is the point G, and the three lines which are not parallel and do not intersect at one point are DE, ZH and TK. We want to draw from the point G a straight line so that the ratio of its segment between the two lines DE and HE to its segment between the two lines ZH and TK is as the ratio of A to B. So we draw GE and we make each of the two lines GL and QM parallel to the line ED. Then we make the ratio of S to A as the ratio of QG to GE, the ratio of O to B as the ratio of ZF to FE, the ratio of GL to ZT as the ratio of S to O, and the product of LK by KT equal to the product of LQ by ZT. Finally we draw GK. Then I say that the ratio of DH to HK is as the ratio of A to B.

Proof: Since the product of LK by KT is equal to the product of LQ by ZT, the ratio of KL to LQ is as the ratio of ZT to TK. Therefore, *invertando* the ratio of LK to KQ is as the ratio of TZ to ZK, and the ratio of LK to KQ is as the ratio of GL to MQ, by the similarity of the triangles.

Therefore the ratio of GL to MQ is as the ratio of TZ to ZK. *Alternando* the ratio of GL to TZ is as the ratio of MQ to ZK, and as the ratio of S to O. Therefore, the ratio of QM to ZK is as the ratio of S to O. But, the ratio of EG to GQ, i.e. the ratio of ED to MQ, is as the ratio of A to S, by the similarity of the two triangles. Therefore, *ex æquali*, the ratio of ED to ZK is as the ratio of A to O, and the ratio of ZK to KC, i.e. the ratio of ZF to FE, is as the ratio of O to B. So, also *ex æquali*, the ratio of A to B is as the ratio of ED to KC, and the ratio of ED to KC is as the ratio of DH to HK, by the similarity of the triangles. Therefore the ratio of DH to HK is as the ratio of A to B. Q.E.D.

[Problem 5]

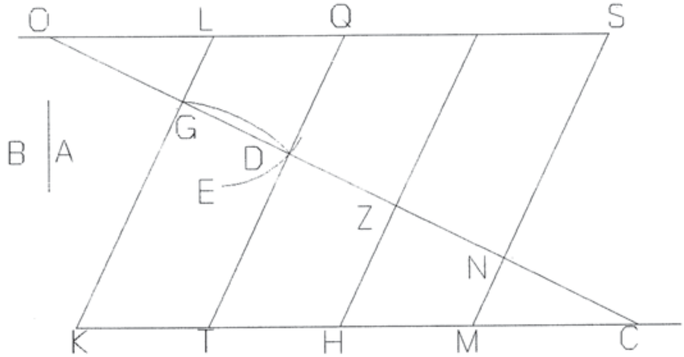


And if the three lines are BG, DE and ZH, and, of them, BG is not straight and DE and ZH are not parallel, and if the known point is the point A, and we want to draw from the point A a straight line so that the ratio of its segment between the two lines BG and DE to its segment between the two lines DE and ZH is known, then according to analysis:

We suppose that the line sought for is ABDZ, that the ratio of BD to DZ is known, and that each of the lines TBK and MAL is parallel to the line DE, so that the ratio of KE to EZ is as the ratio of BD to DZ (which is known). Thus the ratio of KE to EZ is known. And if we [choose H so as to] make this ratio as the ratio of HE to EL, the ratio of HE to EL will be known. But the line EL is known, so the line HE is known. And the point E is known, so the point H is known. So the line HOQ parallel to the line

DE is known in position. Similarly if we make the ratio of MA to AL [equal to] this known ratio itself, the line QTS [through M], which is parallel to the line EK, [is known in position]. And since the ratio of KE to EZ is as HE to EL, the ratio of all of HK to all of LZ is as the known ratio of either antecedent to its consequent. And so the ratio of HK to ZL is known, and it is as the ratio of MA to AL. And the ratio of MA to AL is as the ratio of MS to ZL by the similarity of the two triangles. So the ratio of each of the two lines MS and HK to the line ZL is the same, so the line MS is equal to the line HK. But the line HK is equal to the line QT since the plane [figure] QTKH is a parallelogram, and so the line MS is equal to the line QT. Similarly the line OB is equal to the line AS, by the parallelism of the lines.

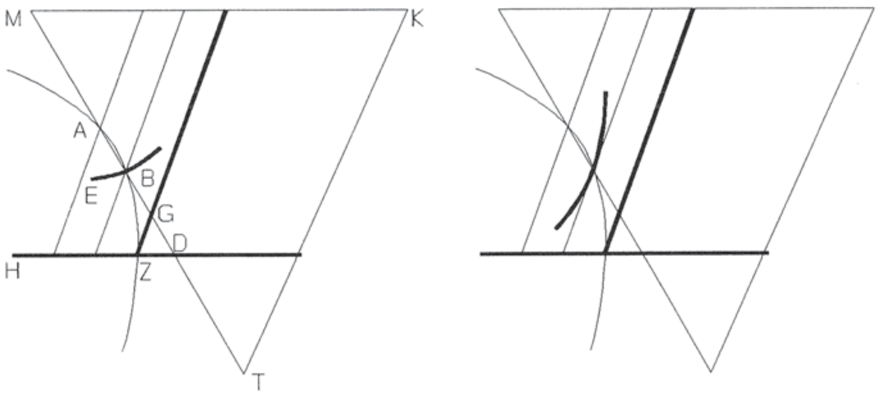
Therefore point B is on the boundary of a hyperbola passing through the point A whose asymptotes are the two lines QH and <QS. And each one of> the two lines QH and QS is known in position, as we proved before. Therefore the boundary of that [hyperbolic] section is known in position, and the line BG is known in position also. So the point B is known since it is on the common section of two lines known in position. And the point A is known, and so the line ABZ may be drawn. Therefore every one of the points Z, D and S is known. Q.E.D.



Synthesis: The known ratio is the ratio of A to B, and the known point is the point G, and the three lines are DE, ZH and HT, of which DE is not straight. We want to draw from the point G a straight line so that the ratio of its segment between the two lines DE and ZH to its segment between the two lines ZH and HT is as the ratio of A to B. So we draw from the point G a line GK parallel to the line ZH and we [choose M so as to] make the ratio of MH to HK as the ratio of A to B, and similarly we [choose L on GK extended so as to] make the ratio of LG to GK [equal to the ratio of A to B]. And we construct the line MS parallel to line LGK and the line SLO [through L] parallel to the line MHK. Then through the point G we construct the hyperbola GD with asymptotes the two lines SM and SO, so it cuts the line DE, say at D. Next we draw the line GD and we produce it in a straight line to the point C. Then I say: The ratio of DZ to ZC is as the ratio of A to B.

Proof: We draw the line QDT parallel to the line ZH. So since the two lines SNM and SLO are asymptotes of the [hyperbolic] section GD the line ND is equal to the line GO, and [therefore, by similarity,] the line SQ is equal to the line LO. But the line SQ is equal to the line MT since the plane [figure] SQTM is a parallelogram. Therefore the line LO is equal to the line MT and so the ratio of each one of the two lines LO and MT to the line CK is the same. And the ratio of LO to CK is as the ratio of LG to GK, which is as the ratio of MH to HK. Therefore the ratio of MT to CK is as the ratio of MH to HK, which is as the ratio of A to B. And this ratio is as the ratio of TH, the remainder, to HC, the remainder. Therefore, the ratio of A to B is as the ratio of TH to HC, and the ratio of TH to HC is as the ratio of DZ to ZC, since the two lines DT and ZH are parallel. Thus the ratio of DZ to ZC is as the ratio of A to B. Q.E.D.

And if we were to make for each of these figures a division of the cases and their limitations, then it would be a big treatise for each one of them, and we hope, God willing, to have the time to do that [someday].



THE SECOND POSITION FOR THE LAST FIGURE FOR THE ANALYSIS

The analysis of the last figure in another way: We want to produce from the known point A a straight line like the line ABGD to the three lines BE, GZ, and DH, of which BE is not straight and GZ and DH are not parallel, so that the ratio of BG to GD is known. According to analysis we suppose that it is known and that it is as the ratio of TG to GA. And so the ratio of TG to GA is known. And so the line which passes through the point T parallel to the line GZ is known in position since the point T is known and the line GZ is known in position. And the ratio of all of TB to all of AD is known, since it is that ratio itself. And if we make this ratio equal to the ratio of MA to AD, then the line which passes through the point M and is parallel to the line DH known in position, and it meets the line TK, since the two of them are not parallel, and so let it meet it at the point K. And since the ratio of each one of the two lines AM and BT to the line AD is the same ratio, then AM is equal to BT. And so the point B is on the boundary of the hyperbola whose asymptotes are the two lines TK and KM, which are known in position. And it passes through the known point A. And so the boundary of that [hyperbolic] section is known in position. And so the point B is known, since it is on the common section between two lines known in position. And so the production of the line ABGD is known. And that is what we wanted to prove.

Notes on the Commentary

In this account of al-Kūhī's treatise we have attempted to find the usual balance between good exposition and faithfulness to the original text. When in doubt we have favoured the latter. Thus, for instance, we tend to avoid the use of algebraic symbolism to represent geometric quantities. Nevertheless, we have occasionally re-ordered statements when the transcription to modern notation made it desirable, particularly when successive members of two series of ratios are found to be equal to each other. In cases such as these, the exposition can be shortened dramatically.

Often al-Kūhī does not state specifically when certain points in the diagram are defined. To clarify the constructions, we introduce the notation (def. X) to indicate an implicit definition of a point.

Al-Kūhī nowhere in this treatise refers to the supporting literature (Euclid's *Elements* and *Data*[11]; Apollonius' *Conics*) when using a known result. We have supplied the references to these treatises. The propositions in Euclid's *Data* used by al-Kūhī in his analyses are given in an appendix to this paper.

General Problem: Given a point and three lines, construct a line through the point so that the two segments formed by cutting this line with the three given lines are to each other in a given ratio. The first three problems deal with what might be considered to be special cases: where at least some of the given lines are parallel, or where the three lines intersect at a point. The fourth problem deals with the general case, three non-parallel lines that do not intersect at a point. Finally, the fifth problem generalizes to allow one of the given lines to be a curve.

The first problem handles the simplest special case, when the three given lines are all parallel.

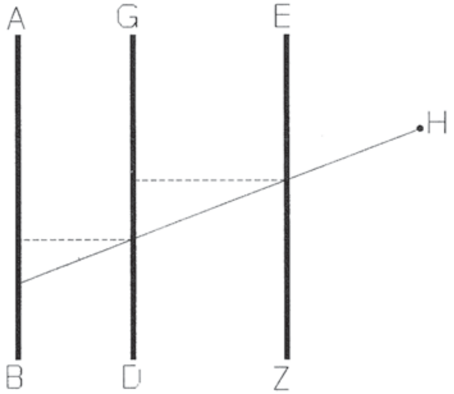
Problem 1(a): Let there be given a point H,[12] three parallel lines (AB,

[11] Euclid's *Data* is available in [8] and [11].

[12] H is not given a letter name in the manuscript.

GD, EZ) and a ratio k equal to the ratio between the two perpendiculars between pairs of adjacent parallels. Then any line drawn through H to the parallel lines is cut into segments by the parallels so that the ratio between the segments is k.

Problem 1(b): As above, except that the given ratio k is not equal to the ratio between the perpendiculars. In this case, no line through the given point cuts the parallel lines in the ratio k.



Proof: None given. The result is taken to be obvious.

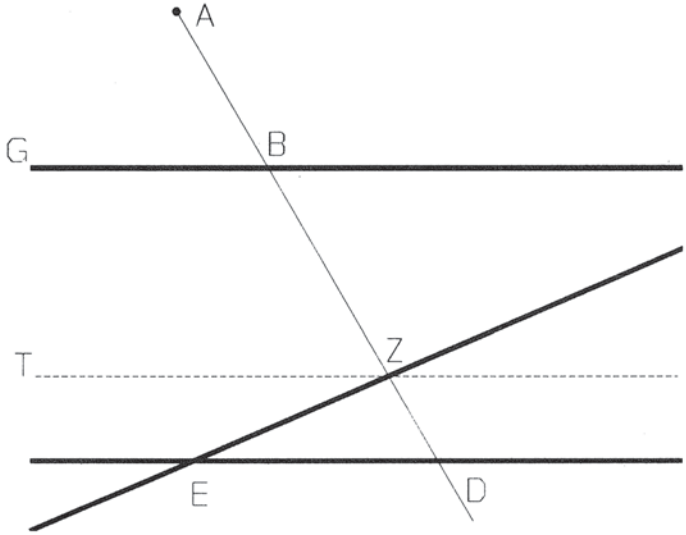
Comments: The point is known in the only sense in which it can be, i.e., in position, although (as al-Kūhī remarks) the position of the point plays the role only of restricting the set of solution lines from all lines not parallel to the given lines to those that pass through P.

Next, al-Kūhī handles the situation where two of the three given lines are parallel.

Problem 2: Given a point and three lines, only two of which are parallel: construct a line through the point that is cut into segments by the given lines according to a given ratio k.

Analysis:

Let A be the given point, BG and DE the parallel lines, and let ZE be the non-parallel line.



Suppose that ABZD is the required line.

Now $\frac{BZ}{DZ}=k$, so $\frac{BD}{DZ}$ is known (*Data*, 6; see commentary).

Also $\frac{AD}{DB}$ is known (*Data,* 34, 6). Thus $\frac{AD}{DZ}$ is known (*Data,* 8).

Thus, the parallel ZT is known in position (can be shown from *Data,* 35, 5, and 8). But the line ZE is known; thus the intersection point Z is known in position (*Data,* 25). Since, then, both A and Z are known, the line AZ is known (*Data,* 26), which was required.

Synthesis:

Given point G, parallel lines DE and TH, a non-parallel line ZH, and ratio α/β:

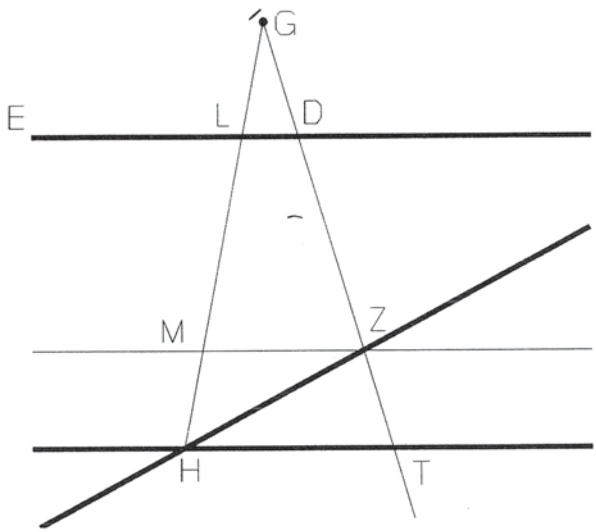
Draw GH (def. L), and construct segment κ[13] so that $\frac{\alpha}{\kappa}=\frac{LH}{GH}$.

Now find M on GH so that $\frac{GH}{MH}=\frac{\kappa}{\beta}$.

Draw through M a line parallel to HT, which intersects the non-parallel line ZH at Z (def. Z).

Draw GZ, and produce it to T (def. D, T). **Claim:** $\frac{DT}{ZT}=\frac{\alpha}{\beta}$.

Proof: $\frac{DT}{TG}=\frac{LH}{HG}=\frac{\alpha}{\kappa}$, and $\frac{GT}{ZT}=\frac{GH}{MH}=\frac{\kappa}{\beta}$. Therefore, *ex aequali*, $\frac{DT}{ZT}=\frac{\alpha}{\beta}$ (*Elements*, V,22). Q.E.D.



[13] Throughout the text, the given ratio and various other ratios are defined as the ratios of two line segments, usually drawn beside the figure. See, for example, the diagrams in the translation. This was a standard practice, especially in medieval Islamic analyses and syntheses. In this commentary we refer to such line segments by Greek letters.

Comments: A series of difficulties present themselves in interpreting the text for this problem. Many of the inferences are wrong and cannot be corrected easily; equally oddly, this is the only problem that seems to pose the given ratio as that between the whole segment cut off by the given lines (DT) and a part (ZT), rather than part-to-part. The synthesis cannot be reconstructed as part-to-part without adding substantial apparatus: the proof by ratio *ex aequali* must commence with the ratio LH:GH, and (since the given point G never appears in the expression of the sought ratio) the term GH must be removed by the *ex aequali* operation. Hence LH, corresponding to the whole segment DT, must remain unless another *ex aequali* is added to the proof.

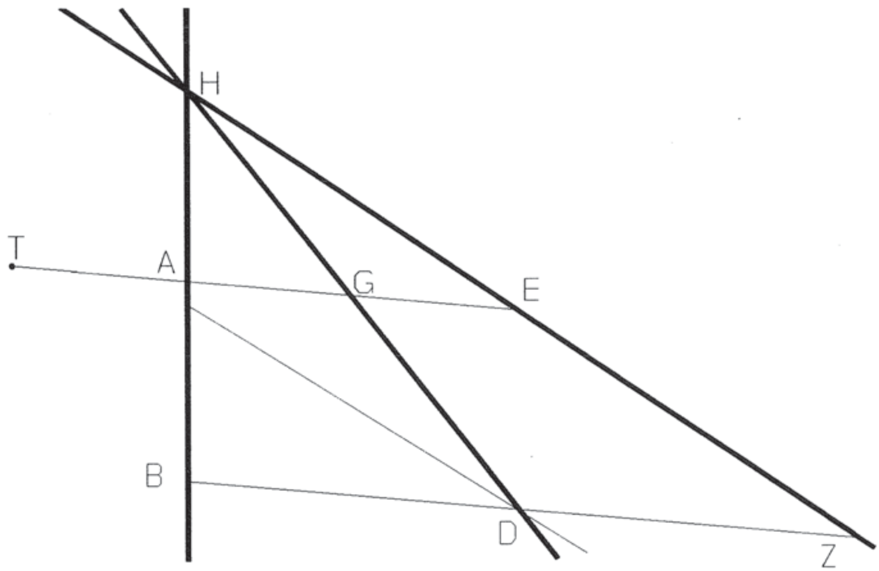
One explanation of this anomaly is as follows: the analysis was originally conceived for the part-to-part problem, and the early statement that BD:DZ is known was actually intended as an inference from the given ratio BZ:ZD. The synthesis may have been composed some time after the analysis (many of al-Kūhī's treatises concluded with promises to complete syntheses or studies of *diorismoi*, time permitting). If so, when returning to the synthesis the author (understandably) may have failed to notice the inference from BZ:ZD to BD:ZD, or realized the triviality of the difference between the whole-to-part and part-to-part problems; he then proceeded to complete a whole-to-part synthesis.[14]

The construction is valid if and only if the following situations do not obtain:

The given ratio α/β is equal to LH/GH. In this case M coincides with G, and the constructed line GZ is parallel to DE and HT.

G lies on the line HT. Then the line GH does not intersect the other parallel, and L is not defined. A solution can, however, be achieved by reversing the roles of the two parallel lines, restating the ratio α/β as $\alpha/(\alpha-\beta)$, and carrying out the original construction.

[14] A couple of line segments stated within the text of the synthesis make no sense under any interpretation, and may be the work of a copyist trying to fix the synthesis to refer to the part-to-part problem.



The final special case is that where the three given lines intersect at a point:

Problem 3: Given a point, and three lines intersecting at another point: construct a line through the former point that is cut into segments by the given lines according to a given ratio *k*.

Analysis:

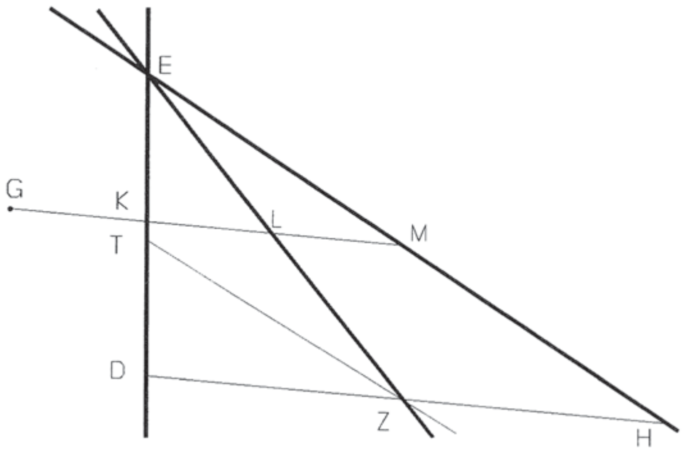
Let T be the given point, and HB, HD, and HZ be the three lines.

Suppose that TAGE is the required line, so that $\frac{AG}{GE}=k$.

Choose point B arbitrarily on HB, and draw BDZ parallel to AGE.

Then $\frac{BD}{AG}=\frac{DH}{HG}=\frac{ZD}{EG}$. Thus $\frac{DB}{DZ}=\frac{AG}{GE}$, the given ratio, so $\frac{BD}{DZ}$ is known.

Since B and $\frac{BD}{ZD}$ are known, the parallel to HZ on which D lies is known in position (*Data,* 35). Since HGD is known, we thus know the intersection point D (*Data,* 25). Therefore line BDZ is known (*Data,* 26), and since T is known and TAGE is parallel to BDZ, we know the required line TAGE as well (*Data,* 28).



Synthesis:

Given point G and lines DE, ZE, and HE:

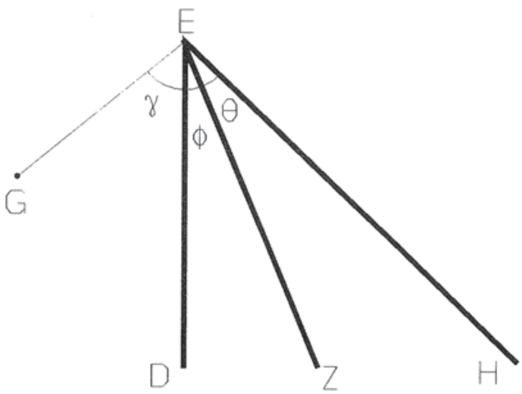
Choose D arbitrarily on DE, and construct T so that $\frac{DT}{TE}=k$.

Draw TZ parallel to EH (def. Z), and draw DZH (def. H).

Then, from G, draw GKLM parallel to DZH (def. K, L, M). **Claim:** $\frac{KL}{LM}=k$.

Proof: $\frac{LK}{ZD}=\frac{LE}{EZ}=\frac{ML}{ZH}$. Thus $\frac{KL}{LM}=\frac{DZ}{ZH}=\frac{DT}{TE}=k$. Q.E.D.

Comments: The given ratio is expressed as α/β, as in other propositions. Since no manipulation of the ratio is required for this problem, we represent it by k.

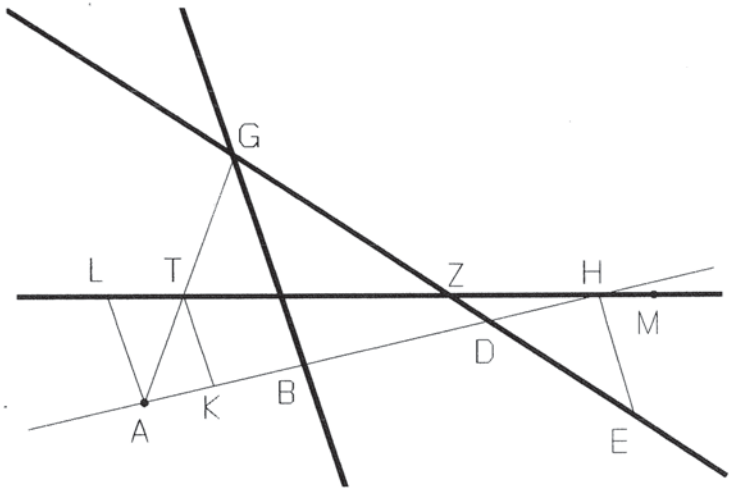


Al-Kūhī's construction works in almost all cases. It fails only if GKLM passes through E, in which case K, L, and M coincide and no ratio KL/LM is formed. This can occur if G coincides with E (in which case there is no solution to the problem), or if the given ratio

$k=\frac{1+\cot\theta\ \tan(\phi+\gamma)}{\cot\phi\ \tan(\phi+\gamma)-1}$, where ϕ, θ, γ are the three angles at right.

Finally, al-Kūhī proceeds to the general case, where the given lines are neither parallel nor intersect at a single point.

Problem 4: Given a point and three non-parallel lines not intersecting at a point: construct a line through the point that is cut into segments by the given lines according to a given ratio $k = \alpha/\beta$.



Analysis:

Let A be the given point, and GB, DE, and ZH be the three given lines. Suppose that ABDH is the required line; then $\frac{BD}{DH}=k$, the given ratio.

Draw HE parallel to BG (def. E); then $\frac{BG}{HE}=\frac{BD}{DH}$ is known.

But $\frac{BG}{HE}=\frac{BG}{ZH}\cdot\frac{ZH}{HE}$, and $\frac{ZH}{HE}$ is known (*Data,* 40, since ΔZHE has the same angles as the triangle formed by the three given lines). Therefore $\frac{BG}{ZH}$ is known.[15]

Draw AG (def. T, which is then known by *Data* 25), and draw TK parallel to BG (def. K). Then $\frac{TA}{AG}$ is known (*Data,* 26 and 1), and

[15] Al-Kūhī's justification by composition of ratios is puzzling, since this can be shown directly by *Data,* 8.

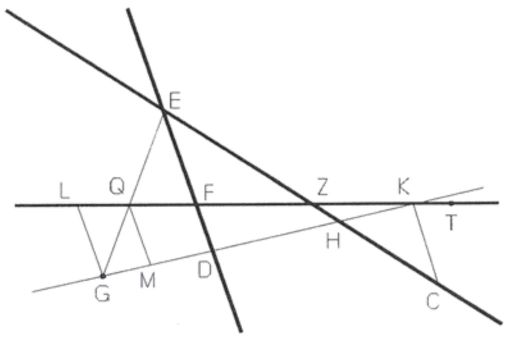
$\frac{TA}{AG}=\frac{TK}{GB}$. Thus, combining the latter ratio with the known ratio $\frac{GB}{ZH}$, $\frac{TK}{ZH}$ is known (*Data,* 8).

Draw AL parallel to BG (def. L),[16] and define M so that $\frac{AL}{ZM}=\frac{TK}{ZH}$.

Since the latter ratio and AL are known (*Data,* 28), ZM is known as well (*Data,* 2); and since Z is known, M is known (*Data,* 27).

Now $\frac{ZH}{ZM}=\frac{TK}{AL}=\frac{TH}{HL}$. From *Elements,* V,17,[17] we have $\frac{HM}{MZ}=\frac{TL}{LH}$.

Thus $TL{\cdot}ZM=HL{\cdot}HM$; and since both TL and ZM are known (*Data,* 26 and 25), so is $HL{\cdot}HM$. But LM is known, so H is known (*Data,* 85). Since A is known, the required line AH is known (*Data,* 26).



Synthesis:

Given point G, the lines DE, ZH, and TK with intersection points E, F, and Z, and the ratio α/β:

[16] This instruction is not given in the text; we infer it from the diagram and the requirements of the subsequent argument.

[17] This is equivalent to subtracting both ratios from 1.

Draw GE (def. Q), and draw lines GL (def. L) and QM parallel to ED.[18]

Let $\frac{\gamma}{\alpha}=\frac{QG}{GE}$ (def. γ) and $\frac{o}{\beta}=\frac{ZF}{FE}$ (def. o).

Construct point T so that $\frac{GL}{ZT}=\frac{\gamma}{o}$.[19]

Then construct point K so that $LK{\cdot}KT=LQ{\cdot}ZT$.[20] **(★)** Draw GK (def. M, D, H).

Claim: $\frac{DH}{HK}=\frac{\alpha}{\beta}$.

Proof:

By **(★)**, $\frac{LK}{LQ}=\frac{ZT}{KT}$. This implies that (◇) $\frac{ZT}{ZK}=\frac{LK}{KQ}=\frac{GL}{MQ}$ (the former equality because $ZK=ZT-KT$ and $QK=LK-LQ$; the latter by triangle similarity).

Thus, *alternando*, $\frac{MQ}{ZK}=\frac{GL}{ZT}$, which is equal to $\frac{\gamma}{o}$. But

$\frac{\alpha}{\gamma}=\frac{EG}{GQ}=\frac{ED}{MQ}$; therefore, *ex aequali*, $\frac{\alpha}{o}=\frac{ED}{ZK}$.

But $\frac{o}{\beta}=\frac{ZF}{FE}=\frac{ZK}{KC}$ (where C is defined by constructing KC parallel to

[18] There is no particular reason for choosing ED here rather than EZ. However, although it is possible to produce a successful argument similar to al-Kūhī's with the choice of EZ, it requires certain substantial modifications both to the diagram and to the argument.

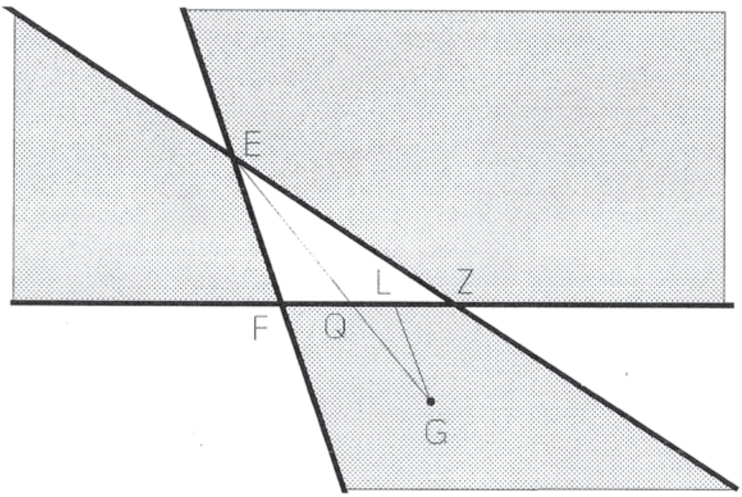
[19] It is implied by the diagram that T should be beyond Z. The proof does not carry through otherwise.

[20] This can be accomplished by *Elements,* VI,28. If one chooses the other possibility for K, another solution is produced. The argument continues to hold with only trivial changes, although the diagram is altered.

ED[21]).

Therefore $\frac{\alpha}{\beta}=\frac{ED}{KC}=\frac{DH}{HK}$. Q.E.D.

Comments: This construction is intended to handle the situation where the three given lines are straight but not parallel and do not share a common intersection point. Thus Problems 1–3 can be regarded as handling the special cases and Problem 4 the general case. Unfortunately the construction given by al-Kūhī applies only to some of the situations involving three non-parallel lines.



The difficulty occurs at (◇), where al-Kūhī asserts that $\frac{LK}{LQ}=\frac{ZT}{KT}$ implies that $\frac{LK}{QK}=\frac{ZT}{ZK}$. In the diagram accompanying the text this is certainly true, and the construction of K (at (★), equivalent to solving a

[21] The line KC and the point C are not explicitly defined in the text, but appear in the figure and are used in the argument..

quadratic equation) is possible because LQ and ZT are disjoint subsegments of LT. Consider, however, the situation where G is in the lower of the three shaded regions in the diagram above. In this case L lies between Q and Z rather than to the left of Q. This change in the sequencing of the points results in the failure of the implication ($\diamond$), and also opens the possibility that K cannot be constructed at ($\star$).

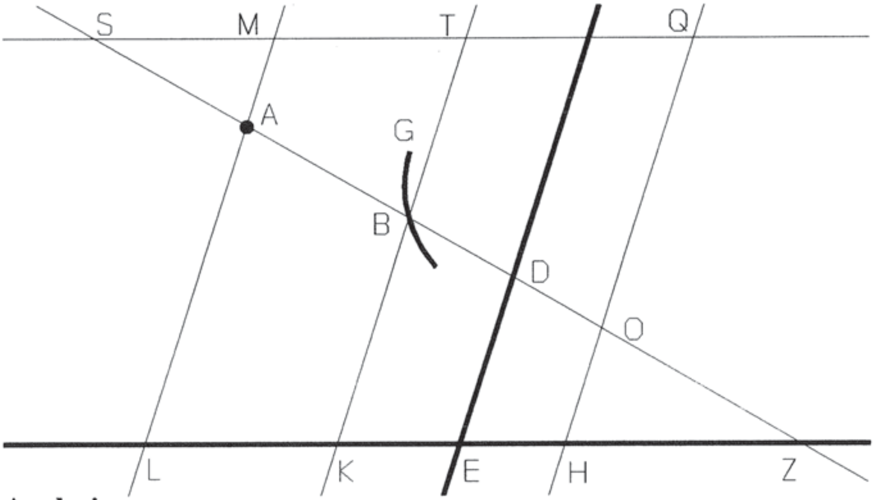
There is some freedom in the construction. The point T can be constructed in either direction along FZ; also, one may change the roles of the three given lines. However, neither of these facts provides a means of repairing al-Kūhī's construction if G lies in one of the three shaded areas in our diagram (those sharing a boundary with one side of the given triangle EFZ).

If G falls on one of the three lines, the construction succeeds provided that the roles of the lines are chosen appropriately so that the constructed line does not coincide with one of the given lines. Some of the ratios of line segments in al-Kūhī's arguments must be restated in this case to avoid division by zero. If G falls on one of the intersection points of the given lines, the problem is either trivially solvable or unsolvable, depending on whether or not the given ratio is zero, but such a case would have been foreign to the ancient or medieval concept of ratio.

The treatise ends with a generalization, allowing one of the three given lines to be a curve. A conic section is required to complete the construction in this case.

Problem 5: Given a point, two non-parallel lines and a curve: construct a line through the point that is cut into segments by the given lines and curve according to a given ratio k.[22]

[22] Again, al-Kūhī refers to the ratio as α/β, but since he does not manipulate the magnitudes α and β, we call the ratio k.



Analysis:

Let the point be A, the curve be BG, and the lines be DE and ZH.

Let ABDZ be the required line; then $\frac{BD}{DZ}=k$.

Draw TBK (def. K) and MAL (def. L), both parallel to DE.

Then $k=\frac{BD}{DZ}=\frac{KE}{EZ}$. Define H so that $(k=)\frac{KE}{EZ}=\frac{HE}{EL}$; since EL and E are known (*Data,* 28, 25 and 26), so are HE and H (*Data,* 2 and 27).

Draw HOQ parallel to DE (def. O); this line is known in position (*Data,* 28).

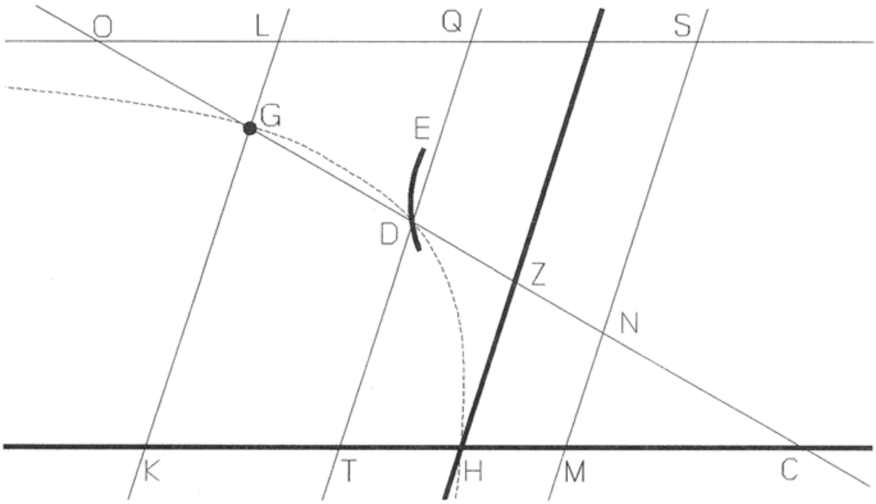
Define M so that $\frac{MA}{AL}=k$, and through M draw QTS parallel to EK (def. Q, T, S). Then QTS is known in position (*Data,* 36).

Now $k=\frac{KE}{EZ}=\frac{HE}{EL}=\frac{KE+HE}{EZ+EL}=\frac{HK}{ZL}$ (*Elements,* V,12), but also $k=\frac{MA}{AL}=\frac{MS}{ZL}$; thus $KH=MS$.

But $HK=QT$; therefore $MS=QT$, which implies that $OB=AS$.

Hence B is on the hyperbola through A with asymptotes QH and QS.[23] Since these asymptotes are known in position and the hyperbola contains the given point A, it is also known in position. But curve BG is known in position; therefore B is known (*Data,* 25).

Since A is known as well, the required line ABZ is known (*Data,* 26).



Synthesis:

Given point G, curve DE, and lines ZH and HT:
Draw GK parallel to ZH (def. K).

Define M (on HK) so that $\frac{MH}{HK}=k$;

similarly, define L (on GK) so that $\frac{LG}{GK}=k$.

Now draw MS parallel to LGK and SL parallel to MHK (def. S).

23 This follows by a straightforward argument from *Conics,* II,4 and II,8. Note that the historical understanding of the hyperbola is as an area; its boundary is equal to the hyperbola in the modern sense.

Through G construct a hyperbola GD with asymptotes SM and SO (*Conics* II.4),[24] and let it cut the curve DE at D (def. D).

Draw GD and produce it to the point C (def. O, Z, N, C). **Claim:**

$$\frac{DZ}{ZC}=k\ .$$

Proof:

Draw QDT parallel to ZH (def. Q, T).

Since SNM and SLO are asymptotes of the hyperbola GD, we know that GO = ND, which implies that LO = SQ(=MT).

Thus $\frac{MT}{CK}=\frac{LO}{CK}=\frac{LG}{GK}=\frac{MH}{HK}=k$, and therefore (by *Elements,* V,19)

$$k=\frac{MT-MH}{CK-HK}=\frac{TH}{HC}=\frac{DZ}{ZC}\quad .\ \text{Q.E.D.}$$

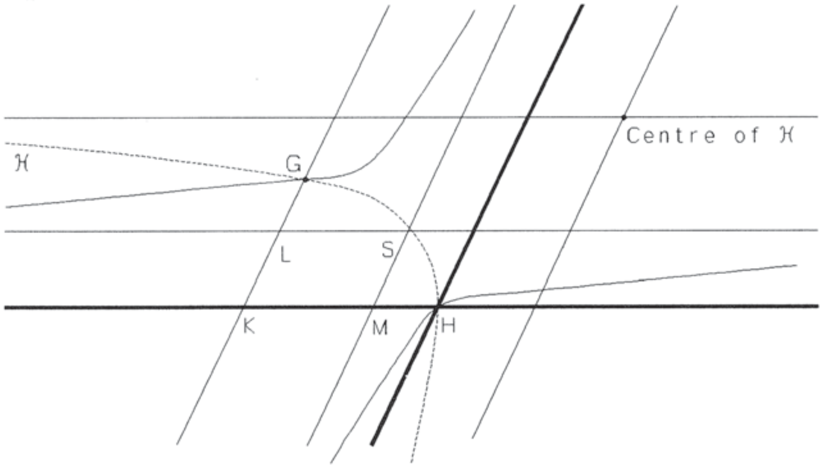
Comments: In this problem, al-Kūhī generalizes the situation further by allowing one of the three lines to be curved (DE). Since this curve has no specified properties, al-Kūhī must proceed almost without referring to it. He works by producing a hyperbola consisting of points P so that, if GP is produced to define Z and C, then $\frac{PZ}{ZC}=k$. Hence, if the hyperbola intersects the given curve, the problem is solved by drawing a line from G through the intersection point. If the hyperbola does not intersect the curve, the problem is implicitly assumed to be unsolvable.

This argument is valid only if the hyperbola H is the entire locus of points L = {P: if GP is produced to Z and C, then $PZ/ZC=k$}. Al-Kūhī shows only that $H\subseteq L$.[25] If there are points P in L but not on H, it is

[24] The point O has not yet been defined. Al-Kūhī could have specified SL instead of SO.

[25] Although the diagram shows only one branch of H, we may assume that H includes the other branch as well, since the argument holds equally well for both branches with a minor modification.

possible that the curve DE passes through one of these points but does not intersect H. In this case al-Kūhī's construction would fail to find a line producing the given ratio k, even though there is a solution.



In fact, H is a proper subset of L. With the aid of Maple (a computer algebra system) we have demonstrated that L is the union of two hyperbolas H and H^*. The second hyperbola H^* is constructed similarly to H, except that the points L and M are drawn within segments GK and HK respectively, rather than beyond the segments as implicitly assumed in the diagram. In this situation al-Kūhī's argument applies equally well.

Note that in the drawing the hyperbolas H and H^* both pass through H, the intersection of the two given lines. It can be shown easily that this must be so. (Since the hyperbola in the diagram in the manuscript is not drawn to pass through H, it is possible that al-Kūhī was not aware that it does.)

Finally, we consider two special cases:

If the given curve happens to pass through G, then G and D coincide and the line GD cannot be drawn. However, the tangent to the hyperbola through G produces the desired ratio.

If the given curve passes through H, then D, Z and C all coincide, and no ratio DZ/ZC is formed.

We do not need to consider cases where D happens to fall on one of the two given lines, since the hyperbola cuts these lines only at H.

Al-Kūhī does not consider any further cases: in particular, that of an arbitrary curve and two parallel lines. The construction of Problem 5 does not work in this situation.

Al-Kūhī's use of a conic section to solve this problem gives us occasion to discuss the role of conics in al-Kūhī's work. Problem 5 is a generalization of Problem 4, which was accomplished strictly according to the principles of plane loci (straight lines and circles; in this case, only straight lines). The inclusion of Problem 4, even though it is made logically redundant by Problem 5, is an indication that al-Kūhī preferred constructions obtained by plane loci to those that appealed to solid loci (as Problem 5 must do). One might argue that Problem 4 is important enough on its own to be worthy of inclusion even though it is superseded by Problem 5, but the structure of a related treatise by al-Kūhī, "On Tangent Circles", also supports our view.[26]

In "On Tangent Circles", al-Kūhī twice solves a problem involving finding a circle whose centre lies on a given line and that is tangent to given lines and circles and/or passes through given points, then generalizes by replacing the given line on which the centre of the circle must lie with a curve (Problems 3 and 4; Problems 6 and 7). In both instances, the later problem renders the earlier one redundant; also, in both cases, the earlier problem does not utilize a conic, whereas the later problem requires it. The inclusion of both problems cannot be due to a distinction between lines and curves: in Problem 1, al-Kūhī solves a problem where the relevant given object is a curve. In this case a straightedge and compass construction suffices to solve the generalized problem, but the corresponding problem where the relevant given object is a line does not appear in the treatise.

At the end of the treatise, after al-Kūhī's usual concluding statement that he hopes to fill in certain special cases of the problem some day, an alternate analysis for Problem 5 appears. It turns out to be virtually identical to the earlier analysis for Problem 5, and is not described here. One wonders whether it was added later by another author.

[26] See Phillippe Abgrall, [1], for an edition and commentary on this work. Our view of the role of conic sections here varies from that presented by Abgrall.

Appendix: Relevant Propositions from Euclid's Data

Since Euclid's *Data* is not accessible to many readers, we repeat the propositions to which al-Kūhī refers, implicitly, in this treatise. The translations of these propositions are taken from [5], replacing Euclid's "given" with al-Kūhī's "known".

1: The ratio of known magnitudes to one another is known.

2: If a known quantity has a known ratio to another quantity, the latter is also known in quantity.

5: If a quantity has a known ratio to a part of it, it will also have a known ratio to the remainder.

6: If two quantities which have a given ratio to one another are conjoined, the whole will also have a given ratio to each of them.

8: Those which have a known ratio to the same, will also have a known ratio to each other.

25: If two curves known in position intersect, the point at which they intersect each other is known.

26: If the extremities of a straight line are known in position, the straight line is known in position and magnitude.

27: If one extremity of a straight line known in position and magnitude is known, the other extremity is also known.

28: If through a known point there is drawn a straight line parallel to a straight line known in position, the straight line so drawn is also known in position.

34: If to parallel straight lines known in position there is drawn a straight line from a known point, that straight line will divided in a known ratio.

35: If from a known point to a straight line known in position there is drawn a straight line, and if it is divided according to a known ratio, and if through the point of intersection there is drawn a straight line parallel to the straight line known in position, that line drawn parallel is known in position.

36: If from a known point there is drawn a straight line to a a straight line known in position, and to it [i.e. the drawn line] a straight line is added having a known ratio to the same, and if through the extremity [of the added line] there is drawn a straight line [parallel] to the line known in position, that line drawn [parallel] is known in position.

40: If each angle of a triangle is known in magnitude, the triangle is known in species.

85: If two straight lines contain a known space in a known angle and the sum of these lines is known, each of these lines will be also known.

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بسم الله الرحمان الرحيم

ثقتي باللـّه سبحانه

رسالة الشيخ الفاضل أبي سهل ويجن بن رستم[1] القوهي في نسبة ما يقع بين ثلاثة خطوط من خطّ واحد.

أمّا أحد الأشكال التي لم تظهر في زمان أحد من الملوك وظهر في زمان مولانا الملك السيّد الأجل شاهانشاه شرف الدولة وزين الملّة، أطال الله بقاءه، وأدام سلطانه، إخراج خطّ مستقيم من نقطة معلومة إلى ثلالثة خطوط الوضع حتّى تكون نسبة أحد القسمين اللذين يقع فيما بين اثنين منها إلى الآخر معلومة.

فلتكن الخطوط الثلاثة اب، جد، هز. فإن كانت هذه الخطوط متوازية، والنسبة المعلومة كنسبة العمود الذي يقع بين هز، جد، إلى العمود الذي يقع بين جد، اب فتكون المسألة سيّالة لأنّ في كلّ خروج من كلّ نقطة تكون هذه النسبة، ولا تتغيّر بتغيّر الخروج ولا بتغيّر[2] النقطة من موضع إلى موضع على وجه من الوجوه.

وإن <لا> كانت النسبة المعلومة كنسبة[3] العمود الذي ذكرناه إلى العمود اللآخر كان من المحال أن يقع بين هذه الخطوط الثلاثة، وهي متوازية، خطّان على تلك النسبة. وهذه شيء ظاهر، أبين ممّا يحتاج إلى برهان، أو شرح أكثر من ذلك. [١٦٥ ب] وذلك ما أردنا أن نبيّن.

[1] رستم - وستم.

[2] بتغيّر - (مطموسة).

[3] كنسبة - (الكاف مطموسة)

وإذا كان خطّان[4] فقط من هذه الخطوط الثلاثة متوازيين، فعلى التحليل ننزل أنّ نقطة ا معلومة، وخطّي ب ج، د ه فقط متوازيان، وخطّي د ه، ز ه يلتقيان على نقطة ه، والخطّ المطلوب ا ب د ز، ونسبة ب د إلى د ز معلومة. ونسبة خطّ ا د[5] إلى خطّ د ب معلومة، لأنّ خطّي ب ج، د ه متوازيان فنسبة خطّ ا د[6] إلى خطّ د ز معلومة فنقطة ز[7] على خطّ مستقيم معلوم الوضع موازٍ لخطّ د ه[8] وهي على خطّ ز ه[9] المعلوم الوضع أيضا، فنقطة ز[10] معلومة، لأنّها على الفصل المشترك[11] بين خطّي ز ه، ز ط[12] المعلومي الوضع. ونقطة ا معلومة، وخطّ ا ب د ز معلوم الوضع، فكلّ واحدة من نقط ا، ب، د، ز معلومة وذلك ما أردنا أن نبيّن.

تركيب ذلك أنّ النسبة المعلومة نسبة ا إلى ب، والنقطة المعلومة نقطة ج، والخطوط الثلاثة د ه، ز ح، ط ح، وخطّي ز ح، ط ح ملتقيان على نقطة ح. فنصل خطّ ج ح فيلقى خطّ د ه على نقطة ل، ونجعل نسبة ا إلى ك كنسبة ل ح إلى ج ح.

4 خطّان - خطين.

5 ا د - ا ج.

6 ا د - ا ج.

7 ز - د.

8 لخطّ د ه - لخطّ ز ه.

9 ز ه - د ه.

10 ز - د.

11 المشترك - المشتركة.

12 ز ه، ز ط - د ه، د ط.

ونجعل نسبة ج ح[13] إلى م ح كنسبة ك إلى ب، ونخرج من نقطة م خطّا موازيا لخطّ ح ط حتى يلقي خطّ ز ح، وليكون لقاؤه على نقطة ز. ونصل خطّ ج ز، ونخرج إلى نقطة ط. فأقول أنّ نسبة د ط[14] إلى ز ط كنسبة آ إلى ب.

برهان ذلك أنّ نسبة د ط[15] إلى ط ج[16] كنسبة ل ح إلى ح ج لأنّ خطّي د ل، ط ح[17] متوازيان. ونسبة ل ح إلى ح ج كنسبة آ إلى ك، فنسبة د ط[18] إلى ط ج[19] كنسبة آ إلى ك. ونسبة ج ط[20] إلى ز ط كنسبة ج ح[21] إلى م ح التي هي كنسبة ك إلى ب. فبالمساواة تكون نسبة د ط[22] إلى ز ط كنسبة آ إلى ب، وذلك ما أردنا أن نبيّن.

وإن كانت الخطوط الثلاثة آ ب، ج د، ه ز غير متوازية، وملتقاها على نقطة واحدة، وهي نقطة ح، ونريد أن نخرج من نقطة ط المعلومة خطّا مستقيما حتى تكون نسبة

13 ج ح - ل م.

14 د ط - د ز.

15 د ط - د ز.

16 ط ج - ز ج.

17 ط ح - ز ح.

18 د ط - د ز.

19 ط ج - ز ج.

20 ج ط - ج ز.

21 ج ح - ج م.

22 د ط - د ز.

ما يقع منه بين خطّي ح ب، ح د[23] إلى ما يقع منه بين حطّي ح د، ح ز معلومة:

فعلى التحليل ننزل أنّ الخطّ المطلوب ط ا ج ه، ونسبة ا ج إلى ج ه معلومة، وأنّ خطّ ب د ز موازٍ لخطّ ا ج ه من نقطة ب المعلومة، فنسبة ب د إلى ا ج كنسبة د ح [١٦٦ ا] إلى ح ج. ونسبة د ح إلى ح ج كنسبة ز د إلى ه ج.

فنسبة د ب إلى ا ج كنسبة ز د إلى ه ج.

وإذا بدّلنا كانت نسبة د ب[24] إلى د ز كنسبة ا ج إلى ج ه. ونسبة ا ج إلى ج ه معلومة. فنسبة ب د إلى د ز معلومة. ونقطة ب معلومة، فنقطة د على خطّ مستقيم معلوم الوضع موازٍ لخطّ ح ز. وهي على خطّ ح ج د المعلوم الوضع. فنقطة د معلومة لأنّها على الفصل المشترك لخطّين معلومي الوضع. ونقطة ب معلومة، فخطّ ب د ز معلوم الوضع. فخطّ ا ج ه معلوم الوضع، لأنّه موازٍ لخطّ ب د ز المعلوم الوضع. فكلّ واحدة من نقطة ا، ج، ه معلومة، وذلك ما أردنا أن نبيّن.

تركيب ذلك أنّ النسبة المعلومة نسبة آ ألى ب، والنقطة المعلومة نقطة ج، والخطوط الثلاث التي ليست بمتوازية، وملتقاها على نقطة واحدة، خطوط د ه، ز ه، ح ه. ونعلّم[25] على خطّ د ه[26] نقطة كيفما اتّفقت، ولتكن نقطة د. ونجعل نسبة د ط إلى ط ه كنسبة آ إلى ب. ونخرج من نقطة ط خطّا موازيا لخطّ ه ح، حتى يلقى خطّ ه ز، وليكن على نقطة ز. ونصل خطّ ز د ح، ونخرج من نقطة ج خطّ ج ك ل

[23] ح د - ح ج.

[24] د ب - ب ز.

[25] ونعلّم - ونتعلم.

[26] على خطّ د ه - على نقطة. (مصحّحة في الهامش).

م موازيا لخطّ د ز ح. فأقول: إنّ نسبة ك ل إلى ل م[27] كنسبة ا إلى ب.

برهان ذلك أنّ نسبة ل ك إلى ز د كنسبة ل ه إلى ه ز، لتشابه المثلّثين. ونسبة ل ه إلى ه ز كنسبة م ل إلى ز ح، فنسبة ل ك إلى ز د كنسبة م ل إلى ح ز.

وإذا بدّلنا كانت نسبة ك ل إلى ل م[28] كنسبة د ز إلى ز ح. ونسبة د ز إلى ز ح كنسبة د ط إلى ط ه لأنّ خطّي ط ز، ه ح متوازيان. فنسبة ك ل إلى ل م[29] كنسبة د ط إلى ط ه. ونسبة د ط إلى ط ه كنسبة ا إلى ب فنسبة ك ل إلى ل م[30] كنسبة ا إلى ب، وذلك ما أردنا أن نبيّن.

وإن كانت الخطوط الثلاثة ب ج، د ه، ز ح[31] غير متوازية، وملتقاها على غير نقطة واحدة، ونريد أن نخرج من نقطة ا المعلومة خطّا حتى تكون نسبة ما يقع منه بين خطّي ب ج، د ه إلى ما يقع منه بين خطّي د ه، ز ح معلومة. فعلى التحليل ننزل أنّ الخطّ المطلوب ا ب د ح، وأنّ نسبة ب د[32] إلى د ح معلومة، وخطّ ح ه موازٍ لخطّ ب ج فنسبة ب ج إلى ه ح تكون معلومة، لأنّها كنسبة ب د إلى د ح المعلومة. [١٦٦ب] لكنّ نسبة ب ج إلى ه ح مؤلّفة من نسبة ب ج إلى ز ح، ومن نسبة ز ح إلى ح ه، ونسبة ز ح إلى ح ه معلومة، لأنّ كلّ واحدة من زوايا مثلّث ز ح ه معلومة، فتبقى نسبة ب ج إلى ز ح معلومة.

27 ل م - ل ه.

28 ل م - ا م.

29 ل م - ا م.

30 ل م - ا م.

31 ز ح - د ح.

32 ب د - ب ه.

وإن وصلنا ا ج، وجعلنا خطّ ط ك موازيا لخطّ ب ج، تكون نسبة ط ا إلى ا ج معلومة. ونسبة ط ا إلى ا ج هي كنسبة ط ك إلى ج ب، فنسبة ط ك إلى ج ب معلومة. وقد كانت نسبة ج ب إلى ز ح معلومة، فنسبة ط ك إلى ز ح معلومة. وإن جعلنا هذه النسبة المعلومة كنسبة خطّ ا ل الموازي لخطّ ب ج، وهو معلوم، إلى خطّ ز م، يكون خطّ ز م معلوما[33]. ونقطة ز معلومة فنقطة م معلومة. وأيضا، لأنّ نسبة ط ك إلى ز ح كنسبة ا ل إلى ز م، فإذا بدّلنا تكون نسبة ط ك إلى ا ل كنسبة ح ز إلى ز م[34]. ولكنّ نسبة ط ك إلى ا ل هي كنسبة ط ح إلى ح ل، فنسبة ط ح إلى ح ل كنسبة ح ز إلى ز م. فإذا قلبنا تكون نسبة ط ل إلى ل ح[35] كنسبة ح م إلى م ز. فضرب ل ط في ز م المعلوم، لأنّ كلّ واحد منهما معلوم، مساوٍ لضرب ل ح في ح م. فضرب ل ح في ح م معلوم. وخطّ ل م معلوم، فنقطة ح معلومة. ونقطة ا معلومة فخطّ ا ب د ح معلوم الوضع. وذلك ما أردنا أن نبيّن.

تركيب ذلك أنّ النسبة المعلومة نسبة ا إلى ب، والنقطة المعلومة نقطة ج، والخطوط الثلاث التي ليست متوازية، ولا ملتقاها على نقطة واحدة: د ه، ز ح، ط ك. ونريد أن نخرج من نقطة ج خطّا مستقيما حتى تكون نسبة ما يقع [منه] بين خطّي د ه، ح ه إلى ما يقع [منه] بين خطّي ز ح، ط ك كنسبة ا إلى ب. فنصل ج ه، ونجعل كلّ واحد من خطّي ج ل، ق م[36] موازيا لخطّ ه د. ونجعل نسبة س[37] إلى ا

33 معلوما - معلوم.

34 ز م - ز د.

35 ل ح - ز ح.

36 ق م - ه م.

37 س - م ز.

كنسبة ق ج[38] إلى ج ه، ونسبة ع إلى ب كنسبة ز ف إلى ف ه. ونجعل نسبة ج ل إلى ز ط كنسبة س إلى ع. ونجعل ضرب ل ك في ك ط مساويا لضرب ل ق[39] في ز ط. ونصل ج ك[40]. فأقول: إنّ نسبة د ح إلى ح ك كنسبة ا إلى ب.

برهان ذلك: من أجل أنّ ضرب ل ك في ك ط مساوٍ لضرب ل ق[41] في ز ط، تكون نسبة ك ل إلى ل ق[42] كنسبة ز ط إلى ط ك. فإذا قلبنا كانت نسبة ل ك إلى ك ق[43] كنسبة ط ز إلى ز ك. ونسبة ل ك إلى ك ق[44] كنسبة ج ل إلى م ق، لتشابه المثلّثين. فنسبة ج ل إلى م ق كنسبة ط ز [١١٦٧] إلى ز ك. وإذا بدّلنا كانت نسبة ج ل إلى ط ز كنسبة م ق إلى ز ك. ونسبة ج ل إلى ط ز كنسبة س إلى ع فنسبة ق م[45] إلى ز ك كنسبة س إلى ع. وقد كانت نسبة ه ج إلى ج ق[46]، أعني نسبة ه د[47] إلى م ق، كنسبة ا إلى س، لتشابه المثلّثين. فبالمساواة تكون نسبة ه د إلى ز ك كنسبة ا إلى ع، ونسبة ز ك إلى ك ص، أعني نسبة ز ف إلى ف ه، كنسبة ع إلى ب. فبالمساواة

38 ق ج - ز ج.

39 ل ق - ل و.

40 ج ك - ج ط.

41 ل ق - ل و.

42 ل ق - ل و.

43 ك ق - ك ر.

44 ك ق - ك و.

45 ق م - ز م.

46 ج ق - ج و.

47 ه د - ه ز.

أيضا، تكون نسبة ا إلى ب كنسبة ه د إلى ك ص. ونسبة ه د إلى ك ص كنسبة د ح إلى ح ك، لتشابه المثلّثين. فنسبة د ح إلى ح ك كنسبة ا إلى ب، وذلك ما أردنا أن نبيّن.

وإن كانت الخطوط الثلاثة ب ج، د ه، ز ح، وب ج منها غير مستقيم، ود ه، ز ح غير متوازيين، والنقطة المعلومة نقطة ا، ونريد أن نخرج من نقطة ا خطّا مستقيما حتى تكون نسبة ما يقع منه بين خطّي ب ج، د ه إلى ما يقع منه بين خطّي د ه، ز ح معلومة، فعلى التحليل: ننزل أنّ الخطّ المطلوب ا ب د ز، ونسبة ب د إلى د ز معلومة، وكلّ واحد من خطّي ط ب ك، م ا ل[48] الموازي لخطّ د ه، حتى تكون نسبة ك ه إلى ه ز كنسبة ب د إلى د ز المعلومة. فنسبة ك ه إلى ه ز معلومة. وإن جعلنا هذه النسبة كنسبة ح ه إلى ه ل تكون نسبة ح ه إلى ه ل معلومة. وخطّ ه ل معلوم فخطّ ح ه معلومة. ونقطة ه معلومة فنقطة ح معلومة. فخطّ ح ع ق الموازي لخطّ د ه معلوم الوضع. وكذلك خطّ ق ط س الموازي لخطّ ه ك إذا جعلنا نسبة م ا[49] إلى ا ل هذه النسبة بعينها المعلومة. ولأنّ نسبة ك ه إلى ه ز[50] كنسبة ح ه إلى ه ل فنسبة جميع ح ك إلى جميع ل ز كنسبة الواحد إلى قرينه المعلومة. فنسبة[51] ح ك إلى ز ل معلومة، وهي كنسبة م ا إلى ا ل. ونسبة م ا إلى ا ل كنسبة م س إلى ز ل[52] لتشابه المثلّثين. فنسبة كلّ واحد من خطّي م س، ح ك إلى خطّ ز ل واحدة، فخطّ م س

48 خطّي ط ب ك، م ا ل - خطوط ط ب، ك م، ا ل.

49 م ا - ب ا.

50 ه ز - (الهاء مطموسة).

51 فنسبة - (مطموسة).

52 ز ل - (اللام مطموسة).

مساوٍ لخطّ ح ك[53]. وخطّ ح ك مساوٍ لخطّ ق ط[54]، لأنّ سطح ق ط ك ح متوازي الأضلاع، فخطّ م س مساوٍ لخطّ ق ط. وكذلك خطّ ع ب مساوٍ لخطّ ا س، لتوازي الخطوط.

فنقطة ب على محيط القطع الزائد المارّ بنقطة ا، ولا يلاقيانه خطّا ق ح، >ق س. وكلّ واحد من<[55] [١٦٧ ب] خطّي ق ح، ق س[56] معلوم الوضع، كما بيّنّا قبل ذلك. فمحيط ذلك القطع معلوم الوضع. وخطّ ب ج معلوم الوضع أيضا. فنقطة ب معلومة لأنّها على الفصل المشترك لخطّين معلومي الوضع. ونقطة ا معلومة، فنخرج خطّ ا ب ز معلوم. فكلّ واحدة من نقط ز، د، س معلومة. وذلك ما أردنا أن نبيّن.

تركيب ذلك أنّ النسبة المعلومة نسبة ا إلى ب، والنقطة المعلومة نقطة ج، والخطوط الثلاثة د ه، ز ح، ح ط[57]، ود ه منها غير مستقيم، ونريد أن نخرج من نقطة ج خطّا مستقيما حتى تكون نسبة ما يقع منه بين خطّي د ه، ز ح إلى ما يقع منه بين خطّي ز ح، ح ط كنسبة ا إلى ب. فنخرج من نقطة ج خطّا موازيا لخطّ ز ح، وليكن ج ك، ونجعل نسبة م ح إلى ح ك كنسبة ا إلى ب، وكذلك نجعل نسبة ل ج إلى ج ك. ونجعل خطّ م س موازيا لخطّ ل ج ك، وخطّ س ل ع موازيا لخطّ م ح ك.

[53] فخطّ م س مساوٍ لخطّ ح ك - لخطّ ح ك.

[54] ق ط - ن ط.

[55] ق س. وكلّ واحد من - (غير مقروءة).

[56] ق ح، ق س - ب ح، س ن.

[57] د ه، ز ح، ح ط - د ه، ز ح، ح ك.

ونعمل على نقطة ج قطع ج د[58] الزائد الذي لا يلقيانه خطـا س م، س ع فيلقى خط د ه، وليكن على نقطة د. ونصل خط ج د، ونخرجه على إستقامة إلى نقطة ص. فأقول: إنّ نسبة د ز إلى ز ص كنسبة ا إلى ب.

برهان ذلك أن نجعل خطّ ق د ط موازيا لخطّ ز ح[59]. فلأنّ خطّي س ن م، س ل ع لا يلتقيان قطع ج د[60] يكون خطّ ن د مساويا لخطّ ج ع[61]، وخطّ س ق مساويا لخطّ ل ع. ولكنّ خطّ س ق مساوٍ لخطّ م ط، لأنّ سطح س ق ط م متوازي الأضلاع. فخطّ ل ع مساوٍ لخطّ م ط، فنسبة كلّ واحد من خطّي ل ع، م ط إلى خطّ ص ك واحدة. ونسبة ل ع إلى ص ك كنسبة ل ج إلى ج ك التي هي كنسبة م ح إلى ح ك. فنسبة م ط إلى ص ك كنسبة م ح إلى ح ك، فنسبة م ط إلى ص ك كنسبة م ح إلى ح ك التي هي كنسبة ا إلى ب. وهذه النسبة هي كنسبة ط ح [١٦٨ ا] الباقي إلى ح ص الباقي. فنسبة ا إلى ب كنسبة ط ح إلى ح ص، ونسبة ط ح إلى ح ص كنسبة د ز إلى ز ص، لأنّ خطّي د ط، ز ح متوازيان. فنسبة د ز إلى ز ص كنسبة ا إلى ب. وذلك ما أردنا أن نبيّن.

ولو جعلنا لهذه الأشكال تقسيم الأوضاع وتحديدها لصار لكلّ واحد منها كتاب كبير. ونحن نرجو التفرّغ إلى عمل ذلك إن شاء الله.

58 قطع ج د - قطع ج د ه.

59 ز ح - د ج.

60 قطع ج د - قطع ج د ه.

61 ج ع - د ع.

الوضع الثاني للشكل الآخر للتحليل

تحليل الشكل الآخر على جهة أخرى: نريد أن نخرج من نقطة ا المعلومة خطّا مستقيما، مثل خطّ ا ب ج د، إلى خطوط ب ه، ج ز، د ح الثلاثة التي ب ه منها غير مستقيم، وج ز، د ح غير متوازيين، حتى تكون نسبة ب ج إلى ج د معلومة. فعلى التحليل ننزل أنّها معلومة، وأنّها كنسبة ط ج إلى ج ا فنسبة ط ج إلى ج ا معلومة فالخطّ الذي يجوز على نقطة ط، موازيا لخطّ ج ز، معلوم الوضع[62] لأنّ نقطة ط[63] معلومة وخطّ ج ز معلوم الوضع. ونسبة جميع ط ب إلى جميع ا د معلومة لأنّها تلك النسبة بعينها. وإن جعلنا هذه النسبة كنسبة م ا إلى ا د يكون الخطّ الذي يجوز على نقطة م، موازيا لخطّ د ح[64] معلوم الوضع. وهو يلقى خطّ ط ك، لأنّهما غير متوازيين، فليلقه على نقطة ك. ولأنّ نسبة كلّ واحد من خطّي ا م، ب ط إلى خطّ ا د نسبة واحدة، فا م مساوٍ لب ط فنقطة ب على محيط القطع الزائد الذي لا يلقيانه خطّا ط ك، ك م المعلوما[65] الوضع. ويمرّ بنقطة ا المعلومة، فمحيط ذلك القطع معلوم الوضع. فنقطة ب[66] معلومة [١٦٨ ب] لأنّها على الفصل المشترك بين خطّين معلومي الوضع. فإخراج خطّ ا ب ج د معلوم. وذلك ما أردنا أن نبيّن.

تمّت الرسالة. والحمد لله سبحانه. وصلّى الله على سيّدنا محمّد وآله وصحبه.

62 الوضع - (غير مقروءة)

63 ط - ا.

64 د ح - د ع.

65 المعلوما - المعلومي.

66 ب - (غير مقروءة).