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HOPFIAN AND CO-HOPFIAN OBJECTS

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Abstract

The aim of the present paper is to study Hopfian and Co-Hopfian objects in categories like the category of rings, the module categories A-mod and mod-A for any ring A. Using Stone's representation theorem any Boolean ring can be regarded as the ring A of clopen subsets of a compact Hausdorff totally disconnected space X. It turns out that the Boolean ring A will be Hopfian (resp. co-Hopfian) if and only if the space X is co-Hopfian (resp. Hopfian) in the category *Top*. For any compact Hausdorff space X let $C_{\mathcal{R}}(X)$(resp. $C_{\mathcal{C}}(X)$) denote the $\mathcal{R}$(resp. $\mathcal{C}$)-algebra of real (resp. complex) valued continuous functions on X. Using Gelfand's representation theorem we will prove that $C_{\mathcal{R}}(X)(C_{\mathcal{C}}(X))$ is Hopfian (respectively co-Hopfian) as an $\mathcal{R}(\mathcal{C})$- algebra if and only if X is co-Hopfian (respectively Hopfian) as an object of *Top*. We also study Hopfian and co-Hopfian compact topological manifolds.

Introduction

The notion of a Hopfian group [4] is by now classical. Throughout the present paper the rings A we consider are associative rings with an identity element $1_A \neq 0$. Any subring B of A is required to satisfy the condition that $1_B = 1_A$. All the modules considered are unitary modules. A-mod (resp. mod-A) will denote the caterogy of left (resp. right) A-modules. In [**12**] V.A. Hiremath has introduced the concept of Hopficity for a ring A regarded as a ring and also for any $M \epsilon A$-mod. We will show in the present paper that A is Hopfian in A-mod if and only if it is Hopfian in mod-A (Theorem 1.3). When these two equivalent conditions are satisfied we will simply say that A is Hopfian as a module. There are obvious dual notions of A being co-Hopfian respectively as a ring, as an object in A-mod and as an object in mod-A. We obtain a necessary and sufficient condition for A to be co-Hopfian in A-mod (Proposition 1.4). Unlike the Hopfian case, by means of a specific

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example we show that co-Hopficity is not left-right symmetric. Also, we will give examples to show that A being Hopfian (resp. co-Hopfian) as a ring and A being Hopfian as a module (resp. co-Hopfian in A-mod or mod-A) are independent of each other. As an immediate consequence of our necessary and sufficient condition it will follow that a not necessarily commutative integral domain A is co-Hopfian in A-mod as well as mod-A if and only if A is a skew-field.

It is a well-known result that any noetherian $M \epsilon A$-mod is Hopfian in A-mod and that any artinian $M \epsilon A$-mod is co-Hopfian ([**17**, page 42]). Arguments used in proving this result will show that any ring A with a.c.c. on two sided ideals is Hopfian as a ring and any ring A with d.c.c on *subrings* is co-Hopfian as a ring. In particular any left noetherian (hence any left artinian) ring A is Hopfian as a ring. Easy examples can be given to show that even fields need not be co-Hopfian as rings. Similar to the result that any left artinian ring is left noetherian we have the result that any ring A which is co-Hopfian in A-mod is automatically Hopfian in A-mod, hence also Hopfian in mod-A (Proposition 1.10). Let n be any integer ≥ 1. It is easy to prove the following implications:

a) $M_n(A)$ Hopfian (resp. co-Hopfian) as a ring $\Rightarrow A$ Hopfian (resp. co-Hopfian) as a ring.
b) $M_n(A)$ Hopfian (resp. co-Hopfian) in $M_n(A)$-mod $\Rightarrow A$ Hopfian (resp. co-Hopfian) in A-mod.

The analogue of Hilbert's basis theorem is valid for Hopficity, namely $M \epsilon A$-mod is Hopfian if and only if $M[X]$ is Hopfian in $A[X]$-mod, where X is an indeterminate over A. This and the analogous result for $M[[X]]$ in $A[[X]]$-mod are proved in Section 2 of the present paper (Theorem 2.1). We do not know whether the analogous result is valid for $M[X, X^{-1}]$ in $A[X, X^{-1}]$-mod. For any non-zero $M \epsilon A$-mod, it is easy to see that $M[X]$ (resp. $M[[X]]$) is not co-Hopfian in $A[X]$ (resp. $A[[X]]$)-mod.

In Section 3 we are mainly concerned with the case when A is commutative. For the results stated in the present paragraph it will be assumed that A is a commutative ring. Then it is well-known [**22**], [**24**] that every f.g. (abbreviation for finitely generated) A-module is Hopfian. It can easily be shown that $M_n(A)$ is Hopfian in $M_n(A)$-mod or mod-$M_n(A)$ for all integers $n \geq 1$. Our necessary and sufficient condition for A to be co-Hopfian in A-mod (Theorem 1.3) implies that A is co-Hopfian in A-mod $\Leftrightarrow A$ is its own total quotient ring. In this case we will prove that $M_n(A)$ is co-Hopfian in both $M_n(A)$-mod and mod-$M_n(A)$. We will also prove that A^n is co-Hopfian in A-mod for all $n \geq 1$. The proof of this will depend on an auxiliary result asserting that an A-homomorphism $f : A^n \to A^n$ is not injective if and only if det f is a zero divisor in A, whatever be the commutative ring A (lemma 3.1). It is also well-known

[2], [25] that every f.g A-module is co-Hopfian if and only if every prime ideal of A is maximal. We will explicitly construct a commutative ring A which is its own total quotient ring admitting a prime ideal which is not maximal. In particular this ring will satisfy the condition that A^n is co-Hopfian in A-mod for each integer $n \geq 1$ and there are f.g A-modules which are not co-Hopfian. The ring A that we construct will have the following additional properties:

c) A is not noetherian
d) A does not have d.c.c for subrings

In [12] Hiremath shows that if the Boolean ring of clopen subsets of a compact Hausdorff totally disconnected space X satisfies the condition that A is Hopfian as a ring then X is co-Hopfian as a topological space. He says he does not know whether the converse to this result is true. Actually we not only show that the converse is true but we also show that A is co-Hopfian as a *ring* if and only if X is Hopfian as a topological space. This is carried out in Section 4.

Let X denote a compact Hausdorff space and $C(X)$ denote either $C_{\mathcal{R}}(X)$ or $C_{\mathcal{C}}(X)$. We regard $C_{\mathcal{R}}(X)$ as an $\mathcal{R}$-algebra and $C_{\mathcal{C}}(X)$ as a $\mathcal{C}$-algebra and simply write "the algebra $C(X)$". Using Gelfand's representation theorem we show that $C(X)$ as an algebra is Hopfian (resp. co-Hopfian) if and only if X is co-Hopfian (resp. Hopfian) in the category *Top* of topological spaces. (Theorem 5.3). We do not have any characterization of compact Hausdorff spaces which are Hopfian (resp. co-Hopfian). However it is an easy consequence of invariance of domain that compact topological manifolds without boundary are co-Hopfian. Among compact manifolds without boundary it can easily be shown that finite sets are the only Hopfian objects. Among compact manifolds with a *non-empty boundary* there are no Hopfian or co-Hopfian objects. If M is a compact manifold with boundary ∂M then the pair $(M, \partial M)$ is a co-Hopfian object in the category Top^2 of pairs of topological spaces.

We conclude our introduction by pointing out that Hilton, Roitberg etc., have studied epimorphisms and monomorphisms in the homotopy category and were led to investigating Hopfian and co-Hopfian objects in the homotopy category [10], [18]. Finally we wish to thank the referee for information on literature. In fact most of the material in Section 7 has been pointed out by the referee.

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1. Hopfian and co-Hopfian rings and modules

Throughout we will formulate our results in the category A-mod of left unital A-modules. There are obvious analogous results in the category mod-A of unital right A-modules. We first fix our terminology and notation. For any $a \epsilon A, \ell_A(a) = \{b \epsilon A \mid ba = 0\}$ and $r_A(a) = \{b \epsilon A \mid ab = 0\}$. By a left (resp. right) zero divisor in A we mean an element $a \neq 0$ in A with $\ell_A(a) \neq 0$ (resp. $r_A(a) \neq 0$). An element $a \epsilon A$ will be called a left (resp. right) unit if there exists an element $c \epsilon A$ with $ca = 1$ (resp. $ac = 1$). We call $a \epsilon A$ left (resp. right) regular if $\ell_A(a) = 0$ (resp. $r_A(a) = 0$). It is trivial to see that any left (resp. right) zero divisor is never a right (resp. left) unit. Also any left regular element a which is a left unit is automatically a two-sided unit.

Definition 1.1. $M \epsilon A$-mod is said to be Hopfian (resp. co-Hopfian) if every surjective (resp. injective) homomorphism $f\colon M \to M$ is an isomorphism.

It is well-known that any noetherian (resp. artinian) module is Hopfian (resp. co-Hopfian) [**17**, Lemma 4, page 41].

Proposition 1.2. *$A \epsilon A$-mod is Hopfian if and only if no left zero divisor in A is a left unit in A. This is theorem 9 in* [**12**]. *Equivalently it is well-known and easy to see that $A \epsilon A$-mod is Hopfian if and only if A is directly finite (i.e. $xy = 1 \Rightarrow yx = 1$)* [**11**].

Theorem 1.3. *$A \epsilon A$-mod is Hopfian if and only if $A \epsilon$ mod-A is Hopfian.*

Proof: Direct finiteness is clearly left right symmetric. ■

Proposition 1.4. *$A \epsilon A$-mod is co-Hopfian if and only if every left regular element $a \epsilon A$ is a two-sided unit.*

Proof: Immediate consequence of the fact that injective homomorphisms $f\colon A \to A$ in A-mod are exactly given by $f(\lambda) = \lambda a$ with $a \epsilon A$ left regular. ■

Examples 1.5. Consider the ring

$$A = \begin{bmatrix} Z/2Z & Z/2Z \\ 0 & Z_{(2)} \end{bmatrix}$$

where $Z_{(2)}$ is the 2-localization of Z, namely $Z_{(2)} = \{\frac{m}{n} \epsilon Q \mid n \text{ odd }\}$. The element $\begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix} \epsilon A$ is easily checked to be right regular but not invertible in A. Hence A is not co-Hopfian in mod-A.

(b) The only ring homomorphism of Z (resp. Q) is the identity map. Hence Z and Q are Hopfian and co-Hopfian as rings. While Z is Hopfian in Z-mod, it is not co-Hopfian in Z-mod. Q is both Hopfian and co-Hopfian in Q-mod (hence also in Z-mod).

(c) For any ring A the unique ring homomorphism φ: $A[X] \to A[X]$ carrying X to X^2 and satisfying $\varphi \mid A = Id_A$ is an injective ring homomorphism which is not surjective. (Here X is an indeterminate over A). Thus $A[X]$ is not co-Hopfian as a ring, whatever be the ring A. A similar argument shows that $A[X, X^{-1}]$ and $A[[X]]$ are not co-Hopfian as rings.

(d) Let $A = K[(X_\alpha)]_{\alpha\epsilon J}$ over a commutative ring K. Then A is a commutative ring hence Hopfian in A-mod. Let Θ: $J \to J$ be a surjective map which is not bijective. Since J is infinite such a map exists. The unique ring homomorphism f: $A \to A$ satisfying $f \mid K = Id_K$ and $f(X_\alpha) = X_{\Theta(\alpha)}$ is then a surjective ring homomorphism which is not an isomorphism. Thus A is not Hopfian as a ring.

(e) Let K be a field and $L = K((X_\alpha)_{\alpha\epsilon J})$ the field of rational functions in an infinite number of indeterminates. Any field is Hopfian as a ring. Thus L is Hopfian as a ring. If Θ: $J \to J$ is any injective map, there is a unique homomorphism φ: $L \to L$ of fields satisfying $\varphi(X_\alpha) = X_{\Theta(\alpha)}$ and $\varphi/K = Id_K$. If Θ is not bijective, then φ is an injective ring homomorphism of L in L which is not surjective. Hence L is not co-Hopfian as a ring. Since L if a field from remarks 1.6(a) and (c) we see that L is both co-Hopfian and Hopfian in L-mod.

(f) For any simple ring A any ring homomorphism f: $A \to B$ is automatically injective. Hence every simple ring is Hopfian as a ring. From example (e) above we see that a simple ring (even a field) need not be co-Hopfian as a ring.

(g) Let K be a field and V an infinite dimensional vector space over K. Let $A = \mathrm{End}_K V$. There exist K-linear surjections f: $V \to V$ which are not injective. Choose such an f. Since $V \xrightarrow{f} V \to 0$ splits in K-mod, $\exists$ a K-linear map h: $V \to V$ with $f \circ h = Id_V$. This means f is a right unit in A. Since ker $f \neq 0$ we can choose a g:$V \to V$ with $g \neq 0$ and $g(V) \subset$ Ker f. Then $g\epsilon A$ satisfies $f \circ g = 0$. Thus f is a right zero divisor in A which is not a right unit in A. From proposition 1.4 we see that A is not Hopfian in mod-A and hence also not in A-mod from theorem 1.3. In case V has countable dimension it follows from exercise 14.13, page 164 of [1] that there are only two non-zero ideals in $A = \mathrm{End}_K V$. Hence A is Hopfian as a ring (see proposition 1.12).

(h) If $A = K[(x_\alpha)_{\alpha\epsilon J}]$ with K any commutative ring and J infinite, from 1.8(d) we see that A is Hopfian in A-mod, but not Hopfian as a ring. When K is a field and V a countably infinite dimensional vector space then $A = \mathrm{End}_K V$ is Hopfian as a ring but not Hopfian as an A-module. As already seen Z is co-Hopfian as a ring but not as a Z-module. When K is a field, $A = K((X_\alpha)_{\alpha\epsilon J})$ the field of rational functions in an infinite number of indeterminates is an example of a commutative ring which is co-Hopfian as a module but not co-Hopfian as a ring from 1.8(e).

Proposition 1.9.

(i) *If A is a ring satisfying a.c.c for two sided ideals then A is Hopfian as a ring.*

(ii) *If A is a ring satisfying d.c.c for subrings then A is co-Hopfian as a ring.*

The proof of this proposition is similar to that of lemma 4, page 42 of [17] and hence omitted.

Proposition 1.10. *Let A be a ring with the property that A is co-Hopfian in A-mod. Then A is automatically Hopfian in A-mod.*

Proof: Let a be a left zero divisor in A. From proposition 1.2 we have only to show that a is not a left unit in A. On the contrary if a is a left unit in A, there exists an element $c\epsilon A$ with $ca = 1$. Then clearly c is left regular. Since A is co-Hopfian in A-mod, from proposition 1.4 we see that c is a two sided unit in A. Then $ca = 1$ implies that a is the inverse of c and hence a is also a two sided unit. This contradicts the fact that a is a left zero divisor. ■

Remarks 1.11. Hiremath [12] has already observed that a direct summand of any Hopfian module is Hopfian. The same observation is valid for co-Hopfian modules as well. He remarks that he does not know of any example of a Hopfian module with a submodule not Hopfian. Later in Section 3 we will construct such modules. Q is Hopfian and co-Hopfian in Z-mod, the quotients Z_{p_∞} of Q are not Hopfian in Z-mod. Later results in Section 3 will also show that quotients of co-Hopfian modules need not be co-Hopfian.

Proposition 1.12. *Let A be a ring and n an integer ≥ 1. Then*

(i) *$M_n(A)$ Hopfian (resp. co-Hopfian) as a ring $\Rightarrow$ A Hopfian (resp. co-Hopfian) as a ring.*

(ii) *$M_n(A)$ Hopfian (resp. co-Hopfian) in $M_n(A)$-mod $\Rightarrow$ A Hopfian (resp. co-Hopfian) in A-mod.*

(iii) *$M_n(A)$ Hopfian (resp. co-Hopfian) in A-mod $\Rightarrow$ A Hopfian (resp. co-Hopfian) in A-mod.*

Proof:

(i) is an immediate consequence of the observations that if $f\colon A \to A$ is a ring homomorphism, $M_n(f)\colon M_n(A) \to M_n(A)$ defined in the obvious way is a ring homomorphism and that $M_n(f)$ is surjective (resp. injective) $\Leftrightarrow$ f is surjective (resp. injective).

(ii) similar to (i) above except for the observation that if $f\colon A \to A$ is a map in A-mod, then $M_n(f)\colon M_n(A) \to M_n(A)$ is a map in $M_n(A)$-mod.

(iii) is immediate from the fact that A is a direct summand of $M_n(A)$ in A-mod. ■

The converses for (ii), (iii) are not true in general. Counter examples will be given in Section 7. But when A is commutative for (ii), (iii) the converses are true and they will be proved in Secton 3. We do not know whether the converse for (i) is true.

Given any non-zero $M \epsilon A$-mod it is known that any infinite direct sum of copies of M is neither Hopfian nor co-Hopfian in A-mod. Any such module will admit the module $N = \oplus_{n\geq 1} M_n$ as direct summand where $M_n = M$ for each $n \geq 1$. The shift map s_+ which carries the nth copy of M to the $(n+1)^{st}$ copy identically is an injective map which is not surjective. The shift map s_- which maps the $(n+1)^{st}$ copy to the n^{th} copy identically for $n \geq 1$ and which maps the 1^{st} copy of M to zero is a surjective map which is not injective. This fact will be made use of by us later in Section 3 for constructing a Hopfian module admitting a non-Hopfian submodule. An infinite direct sum of non-zero modules could very well be simultaneously Hopfian and co-Hopfian. If P denotes the set of all primes, $M = \oplus_{p \epsilon P}(Z/pZ)$ is easily seen to be simultaneously Hopfian and co-Hopfian in Z-mod,

Proposition 1.13. *Let $A[G]$ denote the group ring of a group G over the ring A. If $A[G]$ is Hopfian (resp. co-Hopfian) as a ring then A is Hopfian (resp. co-Hopfian) as a ring and G is Hopfian (resp. co-Hopfian) as a group.*

Proof: Let $f\colon A \to A$ be a homomorphism of rings and $\varphi\colon G \to G$ a homomorphism of groups. Then the map $\beta\colon A[G] \to A[G]$ defined by $\beta\left(\sum_{g \epsilon G} a_g g\right) = \sum_{g \epsilon G} f(a_y)\varphi(g)$ is a ring homomorphism. Also it is easily checked that β is surjective (resp. injective) $\Leftrightarrow$ f and φ are surjective (resp. injective). Proposition 1.13 is an easy consequence of these facts. ■

Remarks 1.14. If $f : A \to A$ is a map in A-mod and $\varphi: G \to G$ is a group homomorphism it is in general not true that $\beta: A[G] \to A[G]$ defined $\beta\left(\sum_{g\in G} a_g g\right) = \sum_{g\in G} f(a_g)\varphi(g)$ will be a map in $A[G]$-mod. In case $\varphi = Id_G$ it is true that β is a map in $A[G]$-mod. The analogue of 1.13 for module categories is not valid. If A is any commutative ring and G *any abelian* group, $A[G]$ is Hopfian in $A[G]$-mod. G need not be Hopfian. However, the following can be proved.

Proposition 1.15. *If $A[G]$ is Hopfian (resp. co-Hopfian) in $A[G]$-mod, then A is Hopfian (resp. co-Hopfian) in A-mod.*

Proposition 1.16. *Let $\{A_\alpha\}_{\alpha\in J}$ be any family of rings and $A = \Pi_{\alpha\in J} A_\alpha$ their direct product.*

(i) *A is Hopfian (resp. co-Hopfian) in A-mod $\Leftrightarrow$ each A_α is Hopfian (resp. co-Hopfian) in A_α-mod.*
(ii) *If A is Hopfian (resp. co-Hopfian) as a ring then each A_α is Hopfian (resp. co-Hopfian) as a ring.*

Proof:

(i) is an immediate consequence of the fact that any map $f: A \to A$ in A-mod is uniquely of the form $\Pi f_\alpha: \Pi A_\alpha \to \Pi A_\alpha$ with $f_\alpha: A_\alpha \to A_\alpha$ a map in A_α-mod and f is surjective (resp. injective) $\Leftrightarrow$ each f_α is surjective (resp. injective).
(ii) If $f_\alpha: A_\alpha \to A_\alpha$ is a ring homomorphism for each $\alpha\in J$, then $f = \Pi f_\alpha: \Pi A_\alpha \to \Pi A_\alpha$ is a ring homomorphism. Moreover f is surjective (resp. injective) if and only if each f_α is surjective (resp. injective). (ii) is an immediate consequence of these facts. ■

Actually proposition 1.16(i) can be improved as follows:

Proposition 1.17. *Let $\{A_\alpha\}_{\alpha\in J}$ be any family of rings and $A = \Pi_{\alpha\in J} A_\alpha$ their direct product. Let $M_\alpha \in A_\alpha$-mod for each $\alpha\in J$. If $M = \Pi_{\alpha\in J} M_\alpha$ with A-action defined by $a.m = (a_\alpha m_\alpha)_{\in J}$ whenever $a = (a_\alpha)_{\alpha\in J}$ with $a_\alpha \in A_\alpha$ and $m = (m_\alpha)_{\alpha\in J}$ with $m_\alpha \in M_\alpha$. Then M is Hopfian (resp. co-Hopfian) in A-mod if and only if each M_α is Hopfian (resp. co-Hopfian) in A_α-mod.*

Again this is an immediate consequence of the fact that any map $f: M \to M$ in A-mod is uniquely of the form $\Pi f_\alpha: \Pi M_\alpha \to \Pi M_\alpha$ with $f_\alpha: M_\alpha \to M_\alpha$ a map in A_α-mod.

2. Hopficity of the modules $M[X], M[X]/(X^n)$ and $M[[X]]$

Given any $M\in A$-mod and an indeterminate X over A we define

$M[X]\epsilon A[X]$-mod, $M[X]/(X^n)\epsilon A[X]/(X^n)$-mod and $M[[X]]\epsilon A[[X]]$-mod as in Section 1 of **[23]**, where $A[X]$ denotes the polynomial ring, $A[X]/(X^n)$ for any integer $n \geq 1$ denotes the truncated polynomial ring and $A[[X]]$ the formal power series ring. The main result proved in this section is the following:

Theorem 2.1. *Let $M\epsilon A$-mod. Then the folowing are equivalent.*

(1) *M is Hopfian in A-mod.*
(2) *$M[X]$ is Hopfian in $A[X]$-mod.*
(3) *$M[X]/(X^n)$ is Hopfian in $A[X]/(X^n)$-mod.*
(4) *$M[[X]]$ is Hopfian in $A[[X]]$-mod.*

Proof: (2) $\Rightarrow$ (1). Let f: $M \rightarrow M$ be any surjective map in A-mod. Then $f[X]$: $M[X] \rightarrow M[X]$ defined by $f[X]\left(\sum_{j=0}^k a_j X^j\right) = \sum_{j=0}^k f(a_j)X^j$ is a surjective map in $A[X]$-mod. Since $M[X]$ is Hopfian in $A[X]$-mod we see that $f[X]$ is injective. This immediately yields the injectivity of f.

The proofs of (3) $\Rightarrow$ (1) and (4) $\Rightarrow$ (1) are similar and omitted.

(1) $\Rightarrow$ (2). Let φ: $M[X] \rightarrow M[X]$ be any surjective $A[X]$-homorphism. Let $\theta = \varphi|M{:}M \rightarrow M[X]$. Then θ is an A-homomorphism: Moreover

$$\varphi\left(\sum_{j=0}^k a_j X^j\right) = \sum_{j=0}^k X^j\theta(a_j). \tag{4}$$

For any $i \geq 0$ let p_i: $M[X] \rightarrow M$ be defined by

$$p_i\left(\sum_{j=0}^k a_j X^j\right) = \begin{cases} a_i & \text{if } i \leq k \\ 0 & \text{if } i > k. \end{cases}$$

Then p_i: $M[X] \rightarrow M$ is a map in A-mod for each $i \geq 0$. Since φ is surjective, given any $c\epsilon M$ there exist an element $\sum_{j=0}^k a_j X^j \in M[X]$ with $\varphi(\sum_{j=0}^k a_j X^j) = c$.

Using 4, we see that the "constant term" of $\theta(a_0)$ is c or equivalently $p_0{\circ}\theta(a_0) = c$. This shows that the map $p_0{\circ}\theta{:}M \rightarrow M$ is a surjective map in A-mod. The Hopfian nature of M in A-mod implies that $p_0{\circ}\theta{:}M \rightarrow M$ is an isomorphism, in particular injective.

Our aim is to show that $\varphi{:}M[X] \rightarrow M[X]$ is injective. Let $\sum_{j=0}^k b_j X^j \in M[X]$ satisfy $\varphi(\sum_{j=0}^k b_j X^j) = 0$. Using 4 and observing that $\theta(b_j) =$

$\sum_{i\geq 0} p_i \circ \theta(b_j)X^i$ we see that $\varphi(\sum_{j=0}^k b_j X^j) = \sum_{j=0}^k d_j X^j +$ terms involving higher powers of X where

$$d_j = p_j \circ \theta(b_0) + p_{j-1} \circ \theta(b_1) + \ldots + p_0 \circ \theta(b_j) \tag{5}$$

for $0 \leq j \leq k$. Hence $\varphi(\sum_{j=0}^k b_j X^j) = 0$ implies $b_j = 0$ for $0 \leq j \leq k$. Writing these out we get the following system of equations:

$$\left.\begin{aligned} p_0 \circ \theta(b_0) &= 0 \\ p_1 \circ \theta(b_0) + p_1 \circ \theta(b_1) &= 0 \\ p_2 \circ \theta(b_0) + p_1 \circ \theta(b_1) + p_0 \circ \theta(b_2) &= 0 \\ \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots & \\ p_k \circ \theta(b_0) + p_{k-1} \circ \theta(b_1) + \ldots \quad \ldots + p_0 \circ \theta(b_k) &= 0 \end{aligned}\right\} \tag{6}$$

We know that $p_0 \circ \theta$ is injective. Hence from the first of these equations we get $b_0 = 0$. Substituting this in the second equation we get $p_0 \circ \theta(b_1) = 0$, hence $b_1 = 0$. Now the third equation will yield $p_0 \circ \theta(b_2) = 0$, hence $b_2 = 0$. Proceeding thus we see that $b_0 = b_1 = \ldots = b_k = 0$. Hence φ is injective.

(1) $\Rightarrow$ (4). The proof is similar to that of (1) $\Rightarrow$ (2). We will only indicate the changes needed. In the proof replace $M[X]$ by $M[[X]]$, $\sum_{j=0}^k a_j X^j$ by $\sum_{j\geq 0} a_j X^j$, equation 4 by $\varphi(\sum_{j\geq 0} a_j X^j) = \sum_{j\geq 0} X^j \theta(a_j)$. The p_i's are defined by $p_i(\sum_{j\geq 0} a_j X^j) = a_i$ for all $i \geq 0$. The calculation of $\varphi(\sum_{j\geq 0} b_j X^j)$ will now be $\varphi(\sum_{j\geq 0} b_j X^j) = \sum_{j\geq 0} d_j X^j$ where $d_j = p_j \circ \theta(b_0) + \ldots + p_0 \circ \theta(b_j)$. Hence $\varphi(\sum_{j\geq 0} b_j X^j) = 0$ if and only if $d_j = 0$ for all $j \geq 0$. The equation $d_j = 0$ combined with the fact that $p_0 \circ \theta: M \to M$ is injective successively yield $b_j = 0$ for all $j \geq 0$. Hence $\varphi: M[[X]] \to M[[X]]$ is injective.

(1) $\Rightarrow$ (3). Again the proof is similar to that of (1) $\Rightarrow$ (2) and hence omitted. ■

Remarks 2.2.

(a) For any $0 \neq M \epsilon A$-mod, the modules $M[X]$ in $A[X]$-mod and $M[[X]]$ in $A[[X]]$-mod are never co-Hopfian. In fact the map "multiplication by X" is an injective non-surjective map in both cases.

(b) If $M \epsilon A$-mod is Hopfian we do not know whether $M[X, X^{-1}]$ will be Hopfian in $A[X, X^{-1}]$-mod. However if $M[X, X^{-1}]$ is Hopfian in $A[X, X^{-1}]$-mod it can be shown that M is Hopfian in A-mod. The proof is similar to the proof (2) $\Rightarrow$ (1) in Theorem 2.1.

3. The Commutative case

Throughout *this section* unless otherwise stated A denotes a commutative ring. The following Lemma might be well-known. As we can not find an explicit reference we include a proof of it here.

Lemma 3.1. *Let P be an $n \times n$ matrix over A. Then there exists a non-zero column vector $\mathbf{a} \epsilon A^n$ with $P\mathbf{a} = \mathbf{0}$ if and only if det $P =$ either 0 or a zero divisor in A, where $\mathbf{0}$ denotes the column vector in A^n with all entries 0.*

Proof: Let $P\mathbf{a} = \mathbf{0}$ with $\mathbf{a} \neq \mathbf{0}$. Let adj P denote the adjugate of P. Then $\mathbf{0} = (\text{adj } P)P\mathbf{a} = (\det P)I_n\mathbf{a} = (\det P)\mathbf{a}$. If a_i is a non-zero entry of $\mathbf{a}$, then $(\det P)a_i = 0$ with $a_i \neq 0$.

Conversely, let $P = d\epsilon A$ and let there exist an element $a \neq 0$ in A with $da = 0$. By induction on n we show that $\exists$ an element $\mathbf{c} \neq \mathbf{0}$ in A^n with $P\mathbf{c} = \mathbf{0}$. For $n = 1$, we have $P = (d)$ and $Pa = 0$ with $a \neq 0$ in A. Assume the result valid for square matrices of size $\leq n - 1$. Let C_{ij} be the i,j cofactor of P. From $P(\text{adj } P)\mathrm{a} = (\det P)\mathrm{a} = \mathbf{0}$ where a is the column vector all of whose entries are a we get $P\begin{pmatrix} C_{11}a \\ \cdots \\ \cdots \\ C_{1n}a \end{pmatrix} = 0$.

If $\begin{pmatrix} C_{11}a \\ \cdots \\ \cdots \\ C_{1n}a \end{pmatrix} \neq \mathbf{0}$, there is nothing to prove. Otherwise $C_{11}a = 0$ and C_{11} is the determinant of the $(n-1) \times (n-1)$ matrix S got from P by deleting the first row and first column. By the inductive assumption $\exists$ an element $\mathbf{v} \neq \mathbf{0}$ in A^{n-1} with $S\mathbf{v} = \mathbf{0}$ in A^{n-1}. If $\mathbf{c} = \binom{0}{\mathbf{v}} \in A^n$, then $\mathbf{c} \neq \mathbf{0}$ in A^n and $P\mathbf{c} = \mathbf{0}$ in A^n. ■

Theorem 3.2. *Let A be a commutative ring with the property that A is co-Hopfian as an A-module. Then for each integer $n \geq 1, A^n$ is co-Hopfian as an A-module.*

Proof: Let $f: A^n \to A^n$ be any injective homomorphism of A-modules. Let P denote the matrix of f w.r.t. the standard basis of A^n. From lemma 3.1 we see that det P is not a zero-divisor in A. From remark 1.6(b) we see that A is its own total quotient ring. It follows that det P is a unit in A, hence P is invertible. This means f is an isomorphism. ■

Examples 3.3. Let B be any abelian group with the property that the p-primary torsion $t_p(B)$ is not zero for every prime p. Let $A = B \oplus Z$ as an abelian group and let us define multiplication in A by $(b, m)(b', m') = (mb' + m'b, mm')$. A is a commutative ring. In fact A is the ring got by adjoining an identity element to B with the so called zero ring structure on B. Every element of the form (b, m) with $m \neq \pm 1$ is a zero-divisor in A. In fact if p is any prime divisor of m, we can choose an element $0 \neq b' \in t_p(B)$ with $pb' = 0$. Then $(b', 0)(b, m) = (0, 0)$. Also from $(b,1)(-b,1) = (0,1)$ and $(b,-1)(-b,-1) = (0,1)$ we see that all the non-zero divisors in A are invertible. From remark 1.6(b) and theorem 3.2 we see that A^n is co-Hopfian in A-mod for every integer $n \geq 1$. B is a two sided ideal in A with $A/B \simeq Z$ as a ring. Thus B is a prime ideal of A. However, B is not maximal, because $\eta^{-1}(mZ) \supsetneq B$ for any $m \neq 0$ where $\eta: A \to A/B$ is the canonical quotient map. Thus from remark 1.6(d) we see that there are finitely generated A-modules which are not co-Hopfian. In fact the cyclic A-module, $A/B \simeq Z$ is itself not co-Hopfian as an A-module. For any $m \neq 0$, multiplication by m is an A-module homomorphism which is injective but not surjective.

In the above example let us choose B as follows. For some prime p_0, let $t_{p_0}(B)$ be an infinite direct sum of copies of Z/p_0Z and for primes $p \neq p_0$ let $t_p(B)$ be any arbitrary p-primary torsion abelian group which is not zero. Let $B = \oplus_{p \in P} t_p(B)$ where P denotes the set of all primes. Since B is an ideal in A, B is an A-submodule of A. The A-endomorphisms of B are the same as the abalian group endomorphisms of B. As an abelian group, $t_{p_0}(B)$ is neither Hopfian nor co-Hopfian. Since $t_{p_0}(B)$ is a direct summand of B, we see that B is neither Hopfian nor co-Hopfian as an abelian group and hence as an A-module. For any subgroup H of B, $H \oplus Z$ is a subring of A. Clearly as an abelian group B does not satisfy the a.c.c. for subgroups. It follows that as a ring, A does not satisfy the a.c.c. for subrings. Since all subgroups of B are ideals in A, it is also clear that A is not noetherian. In this example A is both Hopfian and co-Hopfian as an A-module but the submodule B of A is neither Hopfian nor co-Hopfian.

Proposition 3.4. *For any commutative ring A and any integer $n \geq 1$, the ring $M_n(A)$ is Hopfian as an $M_n(A)$-module.*

Proof: It suffices to prove that $M_n(A)$ is Hopfian in $M_n(A)$-mod. Let X denote a left zero-divisor in $M_n(A)$. We have to show that X is not a left unit in $M_n(A)$. If possible let $Y \in M_n(A)$ satisfy $YX = I_n$. Since A is commutative, it follows that X is invertible in $M_n(A)$ and hence cannot be a left zero divisor contradicting the original assumption. ■

Proposition 3.5. *Let A be a commutative ring which is co-Hopfian in A-mod. Then $M_n(A)$ is co-Hopfian in both the categories $M_n(A)$-mod and mod-$M_n(A)$.*

Proof: Let $X \in M_n(A)$ be a left non-zero divisor in $M_n(A)$. Then we claim that there exists no non-zero row vector $\underline{a} = (a_1, \ldots, a_n)$ in A^n with $\underline{a}\, X = 0 (0 =$ the zero row vector in $A^n)$. Because if there existed such an $\underline{a}$, then the $n \times n$ matrix Y each of whose rows is $\underline{a}$ satisfies

$YX = 0$ and $Y \neq 0 \in M_n(A)$. From the right analogue of lemma 3.1 we see that det X is not a zero divisor in A. But A being co-Hopfian in A-mod, we see that det X is a unit in A, hence X is invertible in $M_n(A)$. Hence $M_n(A)$ is co-Hopfian in $M_n(A)$-mod.

The proof for the other half of the proposition is similar. ■

Let P be a prime ideal in A and $M \epsilon A$-mod. Let M_P be the localization of M regarded as an A_P-module. A natural query is in what way the Hopfian (resp. co-Hopfian) nature of $M \in A$-mod related to the Hopfian (resp. co-Hopfian) nature of $M_P \in A_P$-mod. The following examples show nothing much can be said.

Examples 3.6.

(a) For any prime p, $Z_{p_\infty} \oplus Z$ is neither Hopfian nor co-Hopfian in Z-mod. It's localization at the prime ideal 0 of Z is Q in Q-mod and Q is both Hopfian and co-Hopfian in Q-mod.

(b) For every prime p, let $Z_{(p)} = \{\frac{m}{n} \epsilon Q | (n,p) = 1\}$. Since $Z_{(p)}$ is noetherian as a ring, $Z_{(p)}$ is Hopfian in $Z_{(p)}$-mod, hence Hopfian in Z-mod. If $H = \oplus_{p \epsilon P} Z_{(p)}$, where P is the set of all primes, using the fact that $\mathrm{Hom}_Z(Z_{(p)}, Z_{(q)}) = 0$ if p and q are distinct primes we see that H is Hopfian in Z-mod. Now $Q \otimes H$ is an infinite direct sum of copies of Q and hence not Hopfian in Q-mod.

4. Hopfian and co-Hopfian Boolean rings

Recall that a ring A is said to be Boolean if $a^2 = a$ for all $a \epsilon A$. It is well known that any Boolean ring A is commutative and that $2a = 0$ for any $a \in A$. If A is a Boolean integral domain and $a \neq 0$ in A, then from $(a-1)a = 0$ we see that $a = 1$ and hence $A \simeq Z/2Z$. In particular it follows that any prime ideal in an arbitrary Boolean ring A is necessarily maximal in A. From 1.6(c) and (d) we see that all finitely generated modules over A are Hopfian and co-Hopfian. The object of the present section is to determine necessary and sufficient conditions for A to be Hopfian (resp. co-Hopfian) as a ring using M.H. Stone's representation theorem [20]. Given any compact totally disconnected Hausdorff space X let $B(X)$ denote the set of clopen subsets of X. $B(X)$ turns out to be a Boolean ring under addition and multiplication defined by $C + D = C \nabla D$, the symmetric difference of C and D, and $C \cdot D = C \cap D$. Let H denote $Z/2Z = \{0,1\}$ with the discrete topology. For any set S let $H^S = \Pi_{s \epsilon S} H_s$ where $H_s = H$ for all $s \epsilon S$, endowed with the cartesian product topology. Given a Boolean ring A let $X_A = \{f \in H^A | f(a+b) = f(a) + f(b), f(ab) = f(a)f(b)$ and $f(1) = 1\}$. Then it is known that X_A is a closed subspace of H^A, hence X_A is a compact totally disconnected Hausdorff space. Let $T: A \to B(X_A)$ denote the map $T(a) = \{f \in X_A | f(a) = 1\}$. Then Stone's representation theorem asserts that $T:A \to B(X_A)$ is a ring isomorphism. A nice account of Stone's representation theorem is given in appendix three of [**20**].

Given any continuous map $\varphi:X \to Y$ of compact, Hausdorff totally disconnected spaces, for each clopen set E of $Y, \varphi^{-1}(E)$ is a clopen set of X; also $B(\varphi)$: $B(Y) \to B(X)$ defined by $B(\varphi)(E) = \varphi^{-1}(E) \in B(X)$ for each $E \in B(Y)$ is easily seen to be a ring homomorphism.

Conversely, given a homomorphism $\alpha:A \to B$ of Boolean rings there is an associated map $X(a)$: $X_B \to X_A$. For defining this observe that X_B is nothing but the set of ring homomorphisms of B into $Z/2Z$, regarded as a topological subspace of H^B. Given any ring homomorphism $f:B \to Z/2Z$, clearly $f \circ \alpha:A \to Z/2Z$ is a ring homomorphism. The map $X(\alpha)$ is given by $X(\alpha)(f) = f \circ \alpha$. It turns out to be continuous. Let $B(X(\alpha)):B(X_A) \to B(X_B)$ be the ring homomorphism associated to $X(\alpha)$ as described in the earlier paragraph. Then

$$\begin{array}{ccc} A & \xrightarrow{\alpha} & B \\ {\scriptstyle T}\downarrow & & \downarrow{\scriptstyle T} \\ B(X_A) & \xrightarrow{B(X(\alpha))} & B(X_B) \end{array}$$

diagram 4.1

is known to be a commutative diagram. Actually, for any $a \in A$, we have $T \circ \alpha(a) = \{f:B \to Z/2Z | f$ a ring homomorphism with $f(\alpha(a)) = 1\}$ and $B(X(\alpha)) \circ T(a) = B(X(\alpha))\{g\colon A \to Z/2Z | g$ a ring homomorphism satisfying $g(a) = 1\} = X(\alpha)^{-1}\{g\colon A \to Z/2Z | g$ a ring homomorphism satisfying $g(a) = 1\} = \{f\colon B \to Z/2Z | f$ a ring homomorphism with $(X(\alpha)(f))(a) = 1\} = \{f\colon B \to Z/2Z | f$ a ring homomorphism with $f(\alpha(a)) = 1\} = T \circ \alpha(a)$. This proves the commutativity of diagram 4.1.

Lemma 4.1. *Let X be a compact Hausdorff totally disconnected space and Y a closed subspace of X. Let F be any clopen subset of Y. Then there exists a clopen subset C of X with $C \cap Y = F$.*

This is actually lemma 4 in Hiremath's paper [**12**].

Proposition 4.2. *Let $\alpha:A \to B$ be a homomorphism of Boolean rings. Then*

(i) *α is surjective $\Leftrightarrow$ $X(\alpha):X_B \to X_A$ is injective $\Leftrightarrow$ $B(X(\alpha))$: $B(X_A) \to B(X_B)$ is surjective.*
(ii) *α is injective $\Leftrightarrow$ $X(\alpha)$: $X_B \to X_A$ is surjective $\Leftrightarrow$ $B(X(\alpha))$: $B(X_A) \to B(X_B)$ is injective.*

Proof: (i) From the commutativity of diagram 4.1, we see that α is surjective $\Leftrightarrow B(X(\alpha)): B(X_A) \to B(X_B)$ is surjective.

Assume α surjective. If possible let $X(\alpha)$ be not injective. Let $u \neq v$ be elements in X_B with $X(\alpha)(u) = X(\alpha)(v)$. There exist clopen sets E, F in X_B with $u \in E, v \in F$ and $E \cap F = \phi$. Since $B(X(\alpha))$ is surjective, there exist clopen sets C, D in X_A satisfying $X(\alpha)^{-1}(C) = E$ and $X(\alpha)^{-1}(D) = F$. From $u \in E$ and $v \in F$ we get $X(\alpha)(u) \in C, X(\alpha)(v) \in D$. But we have $X(\alpha)(u) = X(\alpha)(v) = t$ (say). Then $t \in C \cap D$ and hence u and v are in $X(\alpha)^{-1}(C \cap D) = E \cap F$. This contradicts the fact that $E \cap F = \phi$. This proves the implication α surjective $\Rightarrow X(\alpha)$ injective.

Conversely, assume $X(\alpha): X_B \to X_A$ injective. We will prove that $B(X(\alpha)): B(X_A) \to B(X_B)$ is surjective. Since $X(\alpha): X_B \to X_A$ is injective and the spaces involved are compact Hausdorff, it follows that $X(\alpha)$: $X_B \to X(\alpha)(X_B)$ is a homeomorphism when we regard $X(\alpha)(X_B)$ as a subspace of X_A. Given any clopen subset F in X_B, from lemma 4.1 we see that there exists a clopen subset C of X_A with $C \cap X(\alpha)(X_B) = X(\alpha)(F)$, or equivalently $X(\alpha)^{-1}(C) = F$. Thus $B(X(\alpha)): B(X_A) \to B(X_B)$ is surjective, completing the proof of (i).

(ii) Again from the commutativity of diagram 4.1 we see that α is injective $\Leftrightarrow B(X(\alpha))$: $B(X_A) \to B(X_B)$ is injective.

Assume α injective. If possible let $X(\alpha)$: $X_B \to X_A$ be not surjective. Then there exists some $x \in X_A$ with $x \notin X_A - X(\alpha)(X_B)$. Since $X_A - X(\alpha)(X_B)$ is an open set in X_A containing x and since clopen neighbourhoods form a fundamental system of neighbourhoods of any point in X_A, we get a clopen set C in X_A with $x \in C \subset X_A - X(\alpha)(X_B)$. The sets X_A and $X_A - C$ are distinct clopen sets in X_A with $X(\alpha)^{-1}(X_A) = X(\alpha)^{-1}(X_A - C) = X_B$ showing that $B(X(\alpha))$ is not injective. This contradicts the assumption that α is injective.

Conversely, assume that $X(\alpha): X_B \to X_A$ is surjective. If $\varphi: S \to T$ is any set theoretic surjection and $T_1 \neq T_2$ are distinct subsets of T, it is clear that $\varphi^{-1}(T_1) \neq \varphi^{-1}(T_2)$. In particular if C, D are distinct clopen subsets of X_A we see that $X(\alpha)^{-1}(C) \neq X(\alpha)^{-1}(D)$, showing that $B(X(\alpha)): B(X_A) \to B(X_B)$ is injective. This in turn shows that $\alpha: A \to B$ is injective, thus completing the proof of (ii).

Proposition 4.2 implies the well-known result that the isomorphism type of the ring A determines the homeomorphism type of the space X_A. It is well-known **[21]** that $f \to \operatorname{Ker} f$ establishes a bijection between points of X_A and maximal ideals of A. We can transport the topology of X_A to max Spec A using the above bijection. If we start with a compact Hausdorff totally disconnected space X, for each $x \in X$ if we set $I_x = \{C \in B(X) | x \notin C\}$ then $x \mapsto I_x$ is a hoeomorphism of X with max Spec $B(X)$. ■

Definition 4.3. A topological space X is said to be Hopfian (resp. co-Hopfian) in the category *Top* if every surjective (resp. injective) continuous map $f \colon X \to X$ is a homeomorphism.

The main result of this section is the following.

Theorem 4.4. *A Boolean ring A is Hopfian (resp. co-Hopfian) as a ring if and only if X_A is co-Hopfian (resp. Hopfian) in the category Top.*

Proof: Immediate consequence of proposition 4.2. ■

Remarks 4.5. Let J be any finite set. The product space H^J where $H = \{0, 1\}$ is neither Hopfian nor co-Hopfian. As a set H^J is the set of all maps of J into H. For any set theoretic map $\theta \colon J \to J$ we have an induced map $f \mapsto f \circ \theta$ of H^J into H^J, which is easily seen to be continuous. If θ is an injective (resp. surjective) map which is not a bijection, then $f \mapsto f \circ \theta$ is a surjective (resp. injective) map which is not bijective.

Since $A = B(H^J)$ is a commutative ring in which every prime ideal is maximal, all $f \cdot gA$-modules are simultaneously Hopfian and co-Hopfian in the category of A-modules but A is neither Hopfian nor co-Hopfian as a ring.

It would be nice to characterize completely the Hopfian (resp. co-Hopfian) compact Hausdorff totally disconnected spaces.

5. Hopfian and co-Hopfian function algebras

Let K be a commutative ring and K-*alg* denote the category of K-algebras.

Definition 5.1. A K-algebra A is said to be Hopfian (resp. co-Hopfian) as a K-algebra if any surjective (resp. injective) K-algebra homomorphism $f \colon A \to A$ is isomorphism.

Let $\mathcal{R}$ (resp. $\mathcal{C}$) denote the field of real (resp. complex) numbers with the usual topology. For any compact Hausdorff space X let $C_{\mathcal{R}}(X)$ (resp. $C_{\mathcal{C}}(X)$) denote the $\mathcal{R}$ (resp. $\mathcal{C}$)-algebra of continuous functions from X to $\mathcal{R}$ (resp. $\mathcal{C}$). Using the Gelfand respresentation theorem we will determine necessary and sufficient conditions for $C_{\mathcal{R}}(X)$ (resp. $C_{\mathcal{C}}(X)$) to be Hopfian or co-Hopfian in the category $\mathcal{R}$-*alg* (resp. $\mathcal{C}$-*alg*). We will mainly concentrate on $C_{\mathcal{R}}(X)$. Similar results are valid for $C_{\mathcal{C}}(X)$.

X denotes a compact Hausdorff space and $C(X)$ denotes the $\mathcal{R}$-algebra $C_{\mathcal{R}}(X)$. It is well-known that the map $x \mapsto (f(x))_{f \in C(X)}$ is a topological imbedding of X into $\Pi_{f\in C(X)}\mathcal{R}_f$ with the cartesian product topology, where $\mathcal{R}_f = \mathcal{R}$ for each $f \in C(X)$. Also $x \mapsto m_x = \{f \in C(X) | f(x) = 0\}$ is a bijection from X to the set of maximal deals in the $\mathcal{R}$-algebra $C(X)$. If $X \xrightarrow{\varphi} Y$ is a continuous map of compact Hausdorff spaces, there is an induced homomorphism φ^*: $C(Y) \to C(X)$ in $\mathcal{R}$-*alg* given by $\varphi^*(g) = g \circ \varphi$ for every $g \in C(Y)$. Also given any $\mathcal{R}$-algebra homomorphism α: $C(Y) \to C(X)$, there is a unique continuous map $\varphi{:}X \to Y$ satisfying $\alpha = \varphi^*$. To see this, for any $x \in X, \alpha^{-1}(m_x)$ is a maximal ideal of $C(Y)$ and hence $\alpha^{-1}(m_x) = m_{\varphi(x)}$ for a unique element $\varphi(x) \in Y$. If $j_X{:}X \to \Pi_{f\in C(X)}\mathcal{R}_f$ and j_Y: $Y \to \Pi g \in C(Y)\mathcal{R}_g$ denote the imbeddings $j_X(x) = (f(x))_{f\in C(X)}$ and $j_Y(y) = (g(y))_{g\in C(Y)Y}$ respectively, then the set theoretic map φ: $X \to Y$ obtained above satisfies the condition that

$$\begin{array}{ccc} X & \xrightarrow{\varphi} & Y \\ {\scriptstyle j_X}\downarrow & & \downarrow{\scriptstyle j_Y} \\ \prod_{f\in C(X)} \mathcal{R}_f & \xrightarrow{\alpha^*} & \prod_{g\in C(Y)} \mathcal{R}_g \end{array}$$

diagram 5.2

is commutative, where $\alpha^*((r_f)_{f\in C(X)}) = (s_g)_{g\in C(Y)}$ with $s_g = r_{\alpha(g)}$. Since α^* composed with any projection $\Pi\mathcal{R}_g \to \mathcal{R}_g$ is continuous we see that α^* is continuous, hence φ is continuous, provided we check the commutativity of the diagram 5.2. But it is straightforward. Thus the set of $\mathcal{R}$-algebra homomorphisms $C(Y) \to C(X)$ is the same as the set $\{\varphi^*{:}|\varphi{:}X \to Y \text{ continuous }\}$. The results quoted so far are well-known ([**20**, pages 327-330]).

Proposition 5.2. *Let $\varphi{:}X \to Y$ be a continuous map of compact Hausdorff spaces. Then*

(i) *$\varphi^*{:}C(Y) \to C(X)$ is injective $\Leftrightarrow \varphi{:}X \to Y$ is surjective.*
(ii) *$\varphi^*{:}C(Y) \to C(X)$ is surjective $\Leftrightarrow \varphi{:}X \to Y$ is injective.*

Proof:

(i) Suppose $\varphi{:}X \to Y$ is not surjective. Then $\varphi(X)$ is a proper closed subset of Y. We can pick an element $b \in Y - \varphi(X)$. Let $h{:}\varphi(X) \to \mathcal{R}$ be any continuous function. Then we can get continuous extensions $g_1{:}Y \to \mathcal{R}$, $g_2{:}Y \to \mathcal{R}$ of h with $g_1(b) = 0$ and $g_2(b) = 1$ (by Tietze extension theorem). Then $g_1 \neq g_2$ in $C(Y)$ but $\varphi^*(g_1) = h \circ \varphi = \varphi^*(g_2)$ since $g_1|\varphi(X) = g_2|\varphi(X) = h$. Thus φ not surjective $\Rightarrow \varphi^*$ not injective or equivalently φ^* injective $\Rightarrow \varphi$ surjective.

If $\varphi : X \to Y$ is surjective, then for any two set theoretic maps $g_1 : Y \to \mathcal{R}$, $g_2 \colon Y \to \mathcal{R}$, we have the implication $g_1 \circ \varphi = g_2 \circ \varphi \Rightarrow g_1 = g_2$. In particular this implication is true with g_1, g_2 in $C(Y)$. This proves (i). (ii) Suppose φ is not injective, say $x_1 \neq x_2$ in X satisfy $\varphi(x_1) = \varphi(x_2)$. Any $f \in C(X)$ of the form $g \circ \varphi$ with $g \in C(Y)$ has to satisfy $f(x_1) = f(x_2)$. However, we do know that $\exists f \in C(X)$ with $f(x_1) = 0$ and $f(x_2) = 1$. Thus $\varphi^* : C(Y) \to C(X)$ is not surjective.

Conversely, assume that φ is injective. Then $\varphi : X \to \varphi(X)$ is a homeomorphism and $\varphi(X)$ is closed in Y. Given any $f \in C(X), h : \varphi(X) \to \mathcal{R}$ defined by $h\varphi(x) = f(x)$ is continuous. By Tietze extension theorem, there exists $g \in C(Y)$ with $g|\varphi(X) = h$. Then $\varphi^*(g) = f$, showing that $\varphi^* : C(Y) \to C(X)$ is surjective. ■

Theorem 5.3. *Let X be a compact Hausdorff space. Then $C(X)$ is Hopfian (resp. co-Hopfian) as an $\mathcal{R}$-algebra if and only if X is co-Hopfian (resp. Hopfian) as a topological space.*

Proof: Immediate consequence of proposition 5.2. ■

6. Hopfian and co-Hopfian objects in *Top* among compact manifolds

For each integer $n \geq 1$ let D^n denote an n-disk. We may take $D^n = \{x \in \mathcal{R}^n | \; \| x \| \leq 1\}$ where $\|x\|$ denotes the usual norm in $\mathcal{R}^n$. By definition D^0 consists of a point. For $n \geq 1$, the map $x \mapsto \frac{1}{2}x$ is a continuous injection which is not a surjection. The map $\theta : D^n \to D^n$ given by

$$\theta(x) = \begin{cases} 2x & \text{for } \|x\| \leq \dfrac{1}{2} \\[2ex] \dfrac{x}{\| x \|} & \text{for } \|x\| \geq \dfrac{1}{2} \end{cases}$$

is a continuous surjection which is not injective. Thus D^n is neither Hopfian nor co-Hopfian for $n \geq 1$. Observe that $\theta : D^n \to D^n$ defined above has the additional property that $\theta | S^{n-1} = Id_{S^{n-1}}$. Let M^n be any compact topological manifold (with or without boundary) of dimension $n \geq 1$. Then imbedding a disk D^n in M^n we can define a continuous surjection $f : M^n \to M^n$ with $f|(M^n - \operatorname{Int} D^n) = Id_{(M^n - \operatorname{Int} D^n)}$ and $f|D^n$ a continuous surjection of D^n with itself satisfying $f|S^{n-1} = Id_{S^{n-1}}$ and $f|D^n$ not injective. It follows that M^n is not Hopfian. Thus we obtain the following.

Proposition 6.1. *The only compact manifolds (with or without) boundary which are Hopfian are finite discrete spaces.*

As usual for any topological space X we denote the set of arcwise connected components of X by $\Pi_0(X)$.

Theorem 6.2. *Let M^n and N^n be a compact topological manifolds of the same dimension $n \geq 1$, both of them without boundary. Suppose further $|\Pi_0(M^n)| = |\Pi^0(N^n)|$. Then any continuous injection $f{:}M \to N$ is a homeomorphism.*

Proof: Since M^n and N^n are compact we see that $|\Pi_0(M^n)|=|\Pi_0(N^n)| < \infty$. Let $\{M_i^n\}_{i=1}^k$ denote the set of connected components of M^n. Each M_i^n is a compact connected manifold without boundary, of dimension n. Hence $f(M_i^n)$ is a compact, connected subset of N^n. Since f is injective we see that $f|M_i^n{:}M_i^n \to f(M_i^n)$ is a homeomorphism. By invariance of domain it follows that $f(M_i^n)$ is open in N^n. Thus $f(M_i^n)$ is open and closed in N^n and also connected. Hence $f(M_i^n)$ is a connected component of N^n. From the injectivity of f it follows that if $i \neq j$, $f(M_i^n)$ and $f(M_j^n)$ are distinct connected components of N^n. Since $|\Pi_0(N^n)| = |\Pi_0(M^n)| = k < \infty$, it follows that $\{f(M_i^n)\}_{i=1}^k$ are all the connected components of N^n, hence $f{:}M^n \to N^n$ is into. From the compact Hausdorff nature of M and N we see that f: $M^n \to N^n$ is a homeormorphism. ■

As an immediate consequence of theorem 6.2 we get

Corollary 6.3. *Any compact manifold M^n without boundary is co-Hopfian in Top.*

Proposition 6.4. *Any compact manifold M^n with a non-empty boundary ∂M is never co-Hopfian in Top.*

Proof: By Morton Brown's collaring theorem, there exists a homeomorphism $\theta{:}\partial M \times [0,1] \to W$ where W is a neighbourhood of ∂M in M, satisfying $\theta(x,0) = x$ for all $x \in \partial M$. Let $f{:}M \to M$ be defined by $f(u) = u$ for all $u \in M - \theta(\partial M \times [0,1))$, $f(\theta(x,t)) = \theta(x, \frac{1+t}{2})$ for all $x \in \partial M$ and $t \in [0,1]$. Then $f{:}M \to M$ is a continuous injection which is not a surjection. ■

Let Top^2 denote the category of pairs of topological spaces.

Definition 6.5. A pair $(X, A) \in Top^2$ is called Hopfian (resp. co-Hopfian) if any surjective (resp. injective) map $f{:}(X, A) \to (X, A)$ of pairs is a homeomorphism.

For any space X let $H_i(X)$ denote the singular homology with integer coefficients.

Theorem 6.6. *Let M, N be compact manifolds with boundary satisfying the following conditions.*

(i) $\dim M = \dim N$

(ii) *rank* $H_0(\partial M) =$ *rank* $H_0(\partial M)$, *rank* $H_0(M) =$ *rank* $H_0(N)$ *and* *rank* $H_0(M, \partial M) =$ *rank* $H_0(N, \partial N)$.

Then any injective continuous map $f:(M, \partial M) \to (N, \partial N)$ is a homeomorphism.

Proof: Let V denote the double $M_+ \cup_{\partial M} M_-$ of M. Let $r =$ rank $H_0(M)$ and $s =$ rank $H_0(M, \partial M)$. If $i{:}\partial M \to M$ denotes the inclusion, then from the exact sequence $H_0(\partial M) \xrightarrow{i^*} H_0(M) \to H_0(M, \partial M) \to 0$ we see that rank Im $i_* = r - s$. From the Mayer-Vietoris sequence $H_0(\partial M) \xrightarrow{(i_+)_*,(i_-)_*} H_0(M_+) \oplus H_0(M_-) \to H_0(V) \to 0$ where $i_+{:}\partial M \to M_+, i_-{:}\partial M \to M_-$ are the respective inclusions, we see that rank $H_0(V)$ $= 2r-$ rank of image $((i_+)_*, (i_-)_*)$. However $\mathrm{Im}((i_+)_*, (i_-)_*)$ is the same as the diagonal subgroup of Im $i_* \oplus$ Im i_*, hence has the same rank as Im i_*. Thus rank $H_0(V) = 2r - (r - s) = r + s$. Similarly if W denotes the doulbe $N_+ \cup_{\partial N} N_-$ we have rank $H_0(W) = r + s$. In particular we get $|\pi_0(\partial M)| =$ rank $H_0(\partial M) =$ rank $H_0(\partial N) = |\pi_0(\partial N)|$ and $|\pi_0(V)| = r + s = |\pi_0(W)|$.

$f|\partial M{:}\partial M \to \partial N$ is an injective continuous map and $|\pi_0(\partial M)| = |\pi_0(\partial N)|$. Hence theorem 6.2 implies that $f|\partial M$ is a homeomorphism. There is a well-defined continuous map $g{:}V \to W$ satisfying $g|M_+{:}M_+ \to N_+$ and $g|M_-{:}M_- \to N_-$ are the same as f. Then g is injective and $|\pi_0(V)| = |\pi_0(W)|$. From theorem 6.2 again we see that g: $V \to W$ is a homeomorphism. It follows immediately that $f{:}(M, \partial M) \to (N, \partial N)$ is a homeomorphism. ∎

Corollary 6.7. *If M is any compact manifold with boundary ∂M then $(M, \partial M)$ is a co-Hopfian object in Top^2.*

Proof: Immediate consequence of theorem 6.6. ∎

Theorem 6.8.

(i) *If M is any compact manifold without boundary then $C(M)$ is Hopfian in $\mathcal{R}$-alg.*

(ii) *If M is any compact topological manifold with a non-empty boundary ∂M, then $C(M)$ is neither Hopfian nor co-Hopfian in $\mathcal{R}$-alg.*

(iii) *If M is a compact manifold, then $C(M)$ is co-Hopfian in $\mathcal{R}$-alg if and only if M is a finite set.*

Proof: Immediate consequence of theorem 5.3, corollary 6.3 and propositions 6.1 and 6.4. ■

7. Some related results and counter examples

Recall that a ring A is said to be left (resp. right) π-regular if given any $a \epsilon A$ there exists an element $b \epsilon A$ and an integer $n \geq 1$ satisfying $a^n = ba^{n+1}$ (resp. $a^n = a^{n+1}b$). F. Dischinger [**7**],[**8**] has shown that π-regularity is left right symmetric. Before Dischinger obtained this result, G. Azumaya [**3**] referred to a ring which is both left and right π-regular as a strongly π-regular ring. By Dischinger's result A is left π-regular if and only if A is right π-regular if and only if A is strongly π-regular. In [**7**], [**8**] Dischinger also obtained the following results:

1. A ring A is strongly π-regular if and only if every cyclic left or right A-module is co-Hopfian.
2. For a ring A the following conditions are equivalent.
   (i) Every finitely generated left A-module is co-Hopfian.
   (ii) Every finitely generated right A-module is co-Hopfian.
   (iii) $M_n(A)$ is strongly π-regular for all integers $n \geq 1$.

   In [**9**] K.R. Goodearl introduced the concept of a left repetitive ring. A ring A is said to be left repetitive if given any $a \epsilon A$ and any $f \cdot g$ left ideal I of A, the left ideal $\sum_{n \geq 0} Ia^n$ is $f \cdot g$. One of the results proved by Goodearl in [**9**] is the following:
3. Every $f \cdot g M \epsilon A$-mod is Hopfian if and only if $M_n(A)$ is left repetitive for all integers $n \geq 1$.

A good report on these questions including new proofs and rew results can be found in [**13**].

Examples 7.1.

(a) It is clear that $M \epsilon A$-mod Hopfian $\Rightarrow$ End $({}_A M)$ directly finite. In [**19**] J.C. Stepherdson gives examples of directly finite A with $M_n(A)$ not directly finite for some integer $n \geq 2$. For any such ring A, we have A Hopfian in A-mod. Also $M_n(A)$ is not Hopfian in $M_n(A)$-mod. Since A^n is a direct summand of $M_k(A)$ in A-mod, whenever $k^2 > n$, we also see that $M_k(A)$ is not hopfian in A-mod whenever $k^2 > n$.

(b) In part C of [**5**] G.M. Bergman constructs for each integer $n > 1$ a ring A with the property that all regular elements in A are invertible, but $M_n(A)$ is not its own classical ring of quotients. In [**13**] P. Menal constructs a ring A which is its own classical quotient ring but $M_n(A)$ is not Ore, hence $M_n(A)$ does not even have a classical ring of quotients. A careful inspection shows that in both these examples the left regular and the right regular elements of A coincide. Hence by proposition 1.4 in our present paper A is co-Hopfian in both A-mod and mod-A. However, $M_n(A)$ is neither co-Hopfian in $M_n(A)$-mod nor co-Hopfian in mod-$M_n(A)$.

(c) In section 5 of [**16**] it is remarked that W.L. May has a method of obtaining an infinite abelian Hopfian group G such that the complex group algebra $\mathcal{C}(G)$ is not Hopfian as a $\mathcal{C}$-algebra, hence not Hopfian as a ring. In this example $\mathcal{C}$ is Hopfian as a ring, G is Hopfian as a group but $\mathcal{C}[G]$ is not Hopfian as a ring.

8. Open problems

1. If A is Hopfian as a ring, is $A[X]$ Hopfian as a ring?
2. If A is co-Hopfian as a ring and G a co-Hopfian group is $A[G]$ co-Hopfian as a ring?
3. If A is Hopfian in A-mod and G a Hopfian group is $A[G]$ Hopfian in $A[G]$-mod?
4. If A is co-Hopfian in A-mod and G a co-Hopfian group is $A[G]$ co-Hopfian in $A[G]$-mod?
5. If A is Hopfian (resp. co-Hopfian) as a ring is it true that $M_n(A)$ is Hopfian (resp. co-Hopfian) as a ring?
6. Characterize the Hopfian (resp. co-Hopfian) objects in *Top* among compact Hausdorff totally disconnected spaces.
7. If $M \in A$-mod is Hopfian is $M[X, X^{-1}]$ Hopfian in $A[X, X^{-1}]$-mod?

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