

PARABOLIC CURVES FOR DIFFEOMORPHISMS IN $\mathbb{C}^2$

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Abstract

We give a simple proof of the existence of parabolic curves for diffeomorphisms in $(\mathbb{C}^2, 0)$ tangent to the identity with isolated fixed point.

1. Introduction

Let F be a diffeomorphism of $(\mathbb{C}^n, 0)$ tangent to the identity. A *parabolic curve* for F is an injective holomorphic map $\varphi: \Omega \rightarrow \mathbb{C}^n$, where Ω is a simply connected domain in $\mathbb{C}$ with $0 \in \partial\Omega$ such that

- (1) φ is continuous at the origin, and $\varphi(0) = 0$.
- (2) $F(\varphi(\Omega)) \subset \varphi(\Omega)$ and $F^{\circ k}(p)$ converges to 0 when $k \rightarrow +\infty$, for $p \in \varphi(\Omega)$.

We say that φ is *tangent* to $[v] \in \mathbb{P}^{n-1}$ if $[\varphi(\zeta)] \rightarrow [v]$ when $\zeta \rightarrow 0$. Let us write $F(z) = z + P_k(z) + P_{k+1}(z) + \dots$, where P_j is a n -dimensional vector of homogeneous polynomials of degree j , and $P_k \neq 0$. A *characteristic direction* for F is a point $[v] \in \mathbb{P}^{n-1}$ such that $P_k(v) = \lambda v$, for some $\lambda \in \mathbb{C}$; it is *nondegenerate* if $\lambda \neq 0$. The integer $\text{ord}(F) := k \geq 2$ is the *tangency order* of F at 0.

The following theorem is analogous to Briot and Bouquet's theorem [3] for diffeomorphisms of $(\mathbb{C}^n, 0)$.

Theorem 1.1 (Hakim [6]). *Let F be a germ of diffeomorphism of $(\mathbb{C}^n, 0)$ tangent to the identity. For any nondegenerate characteristic direction $[v]$ there exist $\text{ord}(F) - 1$ disjoint parabolic curves tangent to $[v]$ at the origin.*

When $n = 2$, Abate proved that the nondegeneracy condition can be dismissed.

2000 *Mathematics Subject Classification.* 32H02, 32H50, 37F99.

Key words. Dynamical systems, discrete dynamics, parabolic curves, exponential map, blow-ups.

The first author was supported by CAPES, Brazil, Process: BEX3083/05-5.

The third author was supported by FPU program, Spain, Process:AP2005-3784.

Theorem 1.2 (Abate [1], [2]). *Let F be a germ of diffeomorphism of $(\mathbb{C}^2, 0)$ tangent to the identity such that 0 is an isolated fixed point. Then there exist $\text{ord}(F) - 1$ disjoint parabolic curves for F at the origin.*

This theorem is analogous to Camacho-Sad's theorem [4] of existence of invariant curves for holomorphic vector fields. We show in this note that the analogy is deep enough to prove Theorem 1.2 in a simple way starting with Hakim's theorem.

2. Exponential operator and blow-up transformation

Let $\widehat{\mathfrak{X}}_2(\mathbb{C}^2, 0)$ be the module of formal vector fields $X = a(x, y)\frac{\partial}{\partial x} + b(x, y)\frac{\partial}{\partial y}$ of order ≥ 2 , i.e., $\min\{\nu(a), \nu(b)\} \geq 2$. We denote by $\widehat{\text{Diff}}_1(\mathbb{C}^2, 0)$ the group of formal diffeomorphisms tangent to the identity $F(x, y) = (x + p(x, y), y + q(x, y))$ where $\min\{\nu(p(x, y)), \nu(q(x, y))\} \geq 2$. Let us denote by $\mathfrak{X}_2(\mathbb{C}^2, 0)$ and by $\text{Diff}_1(\mathbb{C}^2, 0)$ the convergent elements of $\widehat{\mathfrak{X}}_2(\mathbb{C}^2, 0)$ and $\widehat{\text{Diff}}_1(\mathbb{C}^2, 0)$ respectively.

Let $X \in \widehat{\mathfrak{X}}_2(\mathbb{C}^2, 0)$. The exponential operator of X is the application $\exp tX: \mathbb{C}[[x, y]] \rightarrow \mathbb{C}[[x, y, t]]$ defined by the formula

$$\exp tX(g) = \sum_{j=0}^{\infty} \frac{t^j}{j!} X^j(g)$$

where $X^0(g) = g$ and $X^{j+1}(g) = X(X^j(g))$. Note that, since $\nu(X^j(g)) \geq j + \nu(g)$, we can substitute $t = 1$ to get the element $\exp X(g) \in \mathbb{C}[[x, y]]$. Moreover, $\exp tX$ gives a homomorphism of $\mathbb{C}$ -algebras, in particular, we have

$$\exp tX(fg) = \exp tX(f) \exp tX(g).$$

We get also

Proposition 2.1. *The application*

$$\begin{aligned} \text{Exp}: \widehat{\mathfrak{X}}_2(\mathbb{C}^2, 0) &\rightarrow \widehat{\text{Diff}}_1(\mathbb{C}^2, 0) \\ X &\mapsto (\exp X(x), \exp X(y)) \end{aligned}$$

is a bijection.

Proof: Let

$$G(x, y) = \left(x + \sum_{n=2}^{\infty} p_n(x, y), y + \sum_{n=2}^{\infty} q_n(x, y) \right)$$

and

$$X = \sum_{n=2}^{\infty} \left(a_n(x, y) \frac{\partial}{\partial x} + b_n(x, y) \frac{\partial}{\partial y} \right).$$

The identity $\text{Exp}(X) = G$ is equivalent to

$$\begin{aligned} p_{m+1} &= a_{m+1} + HT_{m+1} \left(\sum_{j=2}^m \frac{1}{j!} X_m^j(x) \right) \\ q_{m+1} &= b_{m+1} + HT_{m+1} \left(\sum_{j=2}^m \frac{1}{j!} X_m^j(y) \right), \end{aligned}$$

where $X_m = \sum_{n=2}^m \left(a_n(x, y) \frac{\partial}{\partial x} + b_n(x, y) \frac{\partial}{\partial y} \right)$, and $HT_{m+1}(h)$ is the homogeneous term of h of order $m+1$. These equations determine univocally X if G is given. $\square$

Note that $\text{ord}(G) = \nu(X)$. In general, X may not be convergent for a convergent G . The formal vector field X such that $G = \text{Exp}(X)$ is called the *infinitesimal generator* of G . If $k = \nu(X)$, then $a_k = p_k$ and $b_k = q_k$, thus the characteristic directions of F correspond to the points of the tangent cone of X . Moreover, if $X = fX'$ with $X' \in \hat{\mathfrak{X}}(\mathbb{C}^2, 0)$ and $f \in \mathbb{C}[[x, y]]$ then $\text{Exp}(X)(x, y) = (x + f(x, y)p(x, y), y + f(x, y)q(x, y))$. The converse statement follows by a process similar to the proof of Proposition 2.1. In particular, 0 is an isolated singular point of X if and only if 0 is an isolated fixed point of F .

Now, let $\pi: (M, D) \rightarrow (\mathbb{C}^2, 0)$ be the blow up of $\mathbb{C}^2$ at the origin, where $D = \pi^{-1}(0) = \mathbb{P}^1$, thus each characteristic direction determines a point of D .

Proposition 2.2. *Let $F \in \text{Diff}_1(\mathbb{C}^2, 0)$. There exists a unique germ of diffeomorphism $\tilde{F}$ in (M, D) such that $\pi \circ \tilde{F} = F \circ \pi$ and $\tilde{F}|_D = \text{id}|_D$. Moreover, the germ $\tilde{F}_p$ has order $\geq \text{ord}(F)$ for any characteristic direction $p \in D$ and hence $\tilde{F}_p \in \text{Diff}_1(M, p)$.*

Proof: Let $F(x, y) = (x + p_k(x, y) + \dots, y + q_k(x, y) + \dots)$ where $k = \text{ord}(F) \geq 2$. We have two charts of $M = U_1 \cup U_2$ such that $\pi|_{U_1}: U_1 \rightarrow \mathbb{C}^2$, is defined by $\pi(x, v) = (x, xv)$ and $\pi|_{U_2}: U_2 \rightarrow \mathbb{C}^2$, is defined by $\pi(u, y) = (uy, y)$. We define $\tilde{F}$ in the first chart as

$$\begin{aligned} \tilde{F}(x, v) &= \pi^{-1} \circ F \circ \pi(x, v) = \left(x + p_k(x, xv) + \dots, \frac{vx + q_k(x, xv) + \dots}{x + p_k(x, xv) + \dots} \right) \\ &= (x + x^k(p_k(1, v) + x(\dots)), v + x^{k-1}(q_k(1, v) - vp_k(1, v) + x(\dots))). \end{aligned}$$

Observe that $\tilde{F}(0, v) = (0, v)$, thus any point of the divisor is fixed. Moreover, if $q_k(1, v_0) - v_0 p_k(1, v_0) = 0$ we have $dF(0, v_0) = I$, and thus for any characteristic direction $p = (0, v_0) \in D$, $\text{ord}(\tilde{F}_p) \geq \text{ord}(F)$. $\square$

Proposition 2.3. *Let $X \in \hat{\mathfrak{X}}_2(\mathbb{C}^2, 0)$. Let $\tilde{X}$ be the formal vector field in (M, D) such that $D\pi \cdot \tilde{X} = X \circ \pi$. If p is a point of the tangent cone of X then $\tilde{X}_p \in \hat{\mathfrak{X}}_2(M, p)$.*

Proof: Let $X = a(x, y)\frac{\partial}{\partial x} + b(x, y)\frac{\partial}{\partial y}$ with $a(x, y) = a_k(x, y) + \dots$, $b(x, y) = b_k(x, y) + \dots$ and $k \geq 2$. Let U_1 and U_2 be two charts of $M = U_1 \cup U_2$ as in the proposition above. Then $\tilde{X}$ is given in the chart U_1 by

$$\begin{aligned} \tilde{X}(x, v) &= a(x, xv)\frac{\partial}{\partial x} + \frac{b(x, xv) - va(x, xv)}{x}\frac{\partial}{\partial v} \\ &= x^k(a_k(1, v) + x(\dots))\frac{\partial}{\partial x} + x^{k-1}((b_k(1, v) - va_k(1, v)) + x(\dots))\frac{\partial}{\partial y}. \end{aligned}$$

Now, if $p = (0, v_0) \in D$ is such that $b_k(1, v_0) - v_0 a_k(1, v_0) = 0$, then $\nu_p(a(x, xv)) \geq k$ and $\nu_p\left(\frac{b(x, xv) - va(x, xv)}{x}\right) \geq k$ so $\tilde{X}_p \in \hat{\mathfrak{X}}_2(M, p)$. $\square$

We say that the singular point p is *strictly singular* if any time we write $X = fX'$, then p is a singular point of X' . Note that in the above statement any strictly singular point of $\tilde{X}$ is in the tangent cone of X . Let us also recall that Seidenberg's reduction of singularities [7] is done by blowing-up at strictly singular points.

Lemma 2.4. *Let $F \in \text{Diff}_1(\mathbb{C}^2, 0)$ and $X \in \hat{\mathfrak{X}}_2(\mathbb{C}^2, 0)$ such that $F = \text{Exp}(X)$. Let $\tilde{X}$ be as in the proposition above. Then for any $p \in D$*

$$\tilde{F}_p = \text{Exp}(\tilde{X}_p).$$

Proof: Let $U \simeq \mathbb{C}^2$ be a chart of M such that $\pi|_U: U \rightarrow \mathbb{C}^2$ is defined by $\pi(x, v) = (x, xv)$ and $p \in U \cap D = \{(0, v) \in U\}$ be a point on the divisor. Without loss of generality, applying a linear change of coordinates, we can suppose that $p = (0, 0) \in U$. Since $F(x, y) = \text{Exp}(X) = (\exp X(x), \exp X(y))$, using the definition of $\tilde{F}$, we have

$$\begin{aligned} \tilde{F}(x, v) &= \left(\exp X(x), \frac{\exp X(xv)}{\exp X(x)} \right) = \left(\exp \tilde{X}(x), \frac{\exp \tilde{X}(xv)}{\exp \tilde{X}(x)} \right) \\ &= \left(\exp \tilde{X}(x), \frac{\exp \tilde{X}(x) \exp \tilde{X}(v)}{\exp \tilde{X}(x)} \right) = (\exp \tilde{X}(x), \exp \tilde{X}(v)) \\ &= \text{Exp}(\tilde{X}_p)(x, v). \end{aligned} \quad \square$$

3. Existence of parabolic curves

We need the following formal version of Camacho-Sad's theorem [4] whose proof goes exactly as the original one (see also [5]).

Theorem 3.1 (Camacho and Sad). *Take $X \in \widehat{\mathfrak{X}}_2(\mathbb{C}^2, 0)$ with an isolated singularity at the origin. There is a desingularization morphism $\sigma: (\tilde{M}, \tilde{D}) \rightarrow (\mathbb{C}^2, 0)$ composition of a finite sequence of blow-ups with centers at strictly singular points and a point $p \in \tilde{D}$ satisfying the following property: There are local coordinates (u, v) at p such that $\tilde{D}_p = (u = 0)$ and the transform X^* of X at p is of the form:*

$$X^*(u, v) = u^m \left((\lambda u + u^2(\cdots)) \frac{\partial}{\partial u} + (\mu v + u(\cdots)) \frac{\partial}{\partial v} \right)$$

where $\lambda \neq 0$, $\frac{\mu}{\lambda} \notin \mathbb{Q}_{>0}$ and $m \geq \nu(X) - 1$.

Remark 3.2. The statement above is also valid when X has a dicritical desingularization; we just need to consider one of the infinitely many nondegenerate characteristic directions on a dicritical divisor.

Let us prove Theorem 1.2. Take X the infinitesimal generator of F , and consider X^* and p as in Camacho-Sad's Theorem. By Lemma 2.4 we have

$$F_p^*(u, v) = \text{Exp}(X_p^*) = (u + \lambda u^{m+1} + O(u^{m+2}), v + \mu u^m v + O(u^{m+1}))$$

so F_p^* is a diffeomorphism tangent to the identity, with $[1, 0]$ as a non-degenerate characteristic direction. By Hakim's Theorem, there exist $\text{ord}(F_p^*) - 1$ disjoint parabolic curves $\varphi_j: \Omega_j \rightarrow \tilde{M}$ for F_p^* tangent to the direction $[1, 0]$ at p . Since this direction is transversal to the divisor, it follows that $\overline{\varphi_j(\Omega_j)} \cap \tilde{D} = \{p\}$ and thereby $\sigma \circ \varphi_j$ is also a parabolic curve for F . This ends the proof.

Remark 3.3. In the case $X = x^k X'$ and $S = (x = 0)$ invariant by X' , Camacho-Sad's index of X at 0 along S is exactly Abate's residual index of F at 0 along S . Furthermore, according to J. Cano's proof [5] of Camacho-Sad's theorem, to find the points $p \in \tilde{D}$ that satisfy Camacho-Sad's theorem, it is enough to follow after the first blow up, the singularities with Camacho-Sad's index not in $\mathbb{Q}_{\geq 0}$. Thus, there exist parabolic curves for any characteristic direction of F that gives at the divisor Abate's residual index not in $\mathbb{Q}_{\geq 0}$ (see Corollary 3.1 in [1]).

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Primera versió rebuda el 22 de febrer de 2007,
 darrera versió rebuda el 7 de setembre de 2007.