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ISOPERIMETRIC INEQUALITIES AND DIRICHLET FUNCTIONS OF RIEMANN SURFACES

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Abstract

We prove that if a Riemann surface has a linear isoperimetric inequality and verifies an extra condition of regularity, then there exists a non-constant harmonic function with finite Dirichlet integral in the surface.

We prove too, by an example, that the implication is not true without the condition of regularity.

1. Introduction.

In this paper we study the relationship between linear isoperimetric inequalities and the existence of harmonic functions with finite Dirichlet integral on Riemann surfaces.

By $\mathcal{S}$ we denote a Riemann surface (whose universal covering space is the unit disk Δ) endowed with its Poincaré metric, *i.e.* the metric obtained by projecting the Poincaré metric of the unit disk: $ds = 2(1-|z|^2)^{-1}|dz|$. With this metric, $\mathcal{S}$ is a complete Riemannian manifold with constant curvature -1. The only Riemann surfaces which are left out are the sphere, the plane, the punctured plane and the tori.

We shall say that a Riemann surface $\mathcal{S}$ satisfies a "linear isoperimetric inequality" (LII) if there exists a finite constant $h(\mathcal{S})$ so that for every relatively compact open set G with smooth boundary we have

$$A(G) \leq h(\mathcal{S})\, L(\partial G).$$

Here and from now on, A, L, d and B refer to Poincaré area, length, distance and open ball of $\mathcal{S}$.

There are connections between LII and some conformal invariants on Riemann surfaces: the bottom of the spectrum of the Laplace-Beltrami operator, $b(\mathcal{S})$, and the exponent of convergence $\delta(\mathcal{S})$:

Theorem A. ([**Ch**], [**B**, p. 228], [**FR**]). *A Riemann surface $\mathcal{S}$ satisfies a linear isoperimetric inequality if and only if $b(\mathcal{S}) > 0$. In fact,*

$$\frac{1}{4} \leq b(\mathcal{S})\, h(\mathcal{S})^2 \qquad \text{and} \qquad b(\mathcal{S})\, h(\mathcal{S}) < \frac{3}{2}.$$

The next result is a well known theorem of Elstrodt-Patterson-Sullivan:

Theorem B. [**S**, p. 333]. *A Riemann surface $\mathcal{S}$ satisfies a linear isoperimetric inequality if and only if $\delta(\mathcal{S}) < 1$. In fact,*

$$b(\mathcal{S}) = \begin{cases} \dfrac{1}{4}, & \text{if } 0 \leq \delta(\mathcal{S}) \leq \dfrac{1}{2}, \\ \delta(\mathcal{S})\,(1 - \delta(\mathcal{S})), & \text{if } \dfrac{1}{2} \leq \delta(\mathcal{S}) \leq 1. \end{cases}$$

A theorem of Myrberg [**T**, p. 522] states that if $\delta(\mathcal{S}) < 1$ (if $\mathcal{S}$ satisfies a LII) then $\mathcal{S}$ has a Green's function ($\mathcal{S} \notin O_G$ in the language of classification theory). If $\mathcal{S}$ is a plane domain (in fact, if $\mathcal{S}$ is a surface of almost finite genus [**SN**, p. 193]), $\mathcal{S}$ has Green's function if and only if $\mathcal{S}$ has a non-constant harmonic function with finite Dirichlet integral [**SN**, p. 194] ($\mathcal{S} \notin O_{HD}$ in the language of classification theory).

One would like to understand the relationship between the classes O_{HD} and $\mathcal{B}$ (the Riemann surfaces which do not satisfy a LII). As we have said above, in the case of surfaces of almost finite genus, $O_G = O_{HD} \subset \mathcal{B}$. The inclusion is strict, as it is shown by the example $\mathcal{S}_0 = \Delta \setminus (\cup_{k=1}^{\infty}\{2^{-k}\} \cup \{0\})$: $\mathcal{S}_0 \notin O_G$ because it is a plane domain whose boundary has positive logarithmic capacity [**T**, p. 81]; $\mathcal{S}_0 \in \mathcal{B}$ because $\cup_{k=1}^{\infty}\{2^{-k}\} \cup \{0\}$ is a discrete set with an accumulation point in Δ [**FR**, Theorem 4].

The inclusion $O_{HD} \subset \mathcal{B}$ is true, in general, with an extra hypothesis:

Theorem 1. *Let $\mathcal{S}$ be a Riemann surface which satisfies a linear isoperimetric inequality. If there exists in $\mathcal{S}$ a set of disjoint simple closed curves $\{\gamma_j\}_{j=1}^m$, such that $\mathcal{S} \setminus \cup_j \gamma_j$ contains n connected components of infinite area $\mathcal{S}_1, \dots, \mathcal{S}_n$, then*

$$\dim HD(\mathcal{S}) \geq n.$$

This inequality is the best possible.

Here $HD(\mathcal{S})$ denotes the (real) linear space of harmonic functions in $\mathcal{S}$ with finite Dirichlet integral.

The inclusion $O_{HD} \subset \mathcal{B}$ is not true in the general case, even with the extra customary hypothesis of bounded geometry [**K**], which in our context means that the injectivity radius $\iota(\mathcal{S})$ is positive. $\iota(\mathcal{S})$ is defined as

$$\iota(\mathcal{S}) = \inf\{\iota(p) : \ p \in \mathcal{S}\},$$

where $\iota(p)$ is the injectivity radius of the geodesic exponential map centered at p.

Theorem 2. *There exists a Riemann surface $\mathcal{R} \in O_{HD}$, with $\iota(\mathcal{R}) > 0$, which satisfies a linear isoperimetric inequality.*

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2. Proof of Theorem 1.

Without loss of generality, we can assume that $\{\gamma_j\}_{j=1}^m$ are simple closed geodesics. If this is not the case, we can substitute each curve by the geodesic in its same free homotopy class.

Let $\mathcal{S}_k$ be a component of infinite area of $\mathcal{S} \setminus \cup_j \gamma_j$, and let $\mathcal{S}_k^*$ be the Schottky double of $\mathcal{S}_k$ (see [**AS**, p. 26] for the definition).

Claim. $\mathcal{S}_k^*$ satisfies a linear isoperimetric inequality.

If the claim is true, the theorem of Myrberg [**T**, p. 522] states that $\mathcal{S}_k^*$ has Green's function. This implies that the Royden's harmonic boundary of $\mathcal{S}_k^*$ is not empty [**SN**, p. 166].

$\mathcal{S}_k^*$ is symmetrical with respect to $\partial\mathcal{S}_k$, a compact set which separates $\mathcal{S}_k^*$ in two connected components. Then the Royden's harmonic boundary of S_k^* is also symmetrical with respect to $\partial\mathcal{S}_k$, and contains at least two points, one of them corresponding to $\mathcal{S}_k$ (one of them is in the closure of $\mathcal{S}_k$ in the Royden's compactification of $\mathcal{S}_k^*$).

This is true for $k = 1, \dots, n$. Therefore, the Royden's harmonic boundary of $\mathcal{S}$ contains at least n points [**SN**, p. 191]. This is equivalent [**SN**, p. 166] to

$$\dim HD(\mathcal{S}) \geq n.$$

This inequality is the best possible:

Let $\mathcal{R}$ be the Riemann surface given by Theorem 2 ($\mathcal{R}$ will be constructed without any mention to Theorem 1) and consider $\mathcal{R}_n$, a n-covering of $\mathcal{R}$ based in a closed simple geodesic $\gamma \subset \mathcal{R}$. $\mathcal{R}_n$ satisfies the hypothesis of this theorem and also $\dim HD(\mathcal{R}_n) = n$ (the Royden's harmonic boundary of $\mathcal{R}_n$ consists of n points, because the Royden's harmonic boundary of $\mathcal{R}$ consists of one point ($\mathcal{R} \in O_{HD} \setminus O_G$) [**SN**, p. 166]).

To finish the proof of Theorem 1, we only need to prove the Claim:

By a geodesic domain in a Riemann surface we mean a connected domain G with finite area, such that ∂G consists of finitely many closed simple geodesics. G does not have to be relatively compact since it may "surround" finitely many punctures.

The following lemma will be very useful:

Lemma. [**FR**, p. 168]. *A Riemann surface satisfies a* LII *if and only if it satisfies* LII *for geodesic domains. Moreover, if h and h_g are, respectively, the usual and geodesic isoperimetric constants, then*

$$h_g \le h \le 2 + h_g .$$

Therefore, we must verify LII only for geodesic domains of $\mathcal{S}_k^*$. By the symmetry of $\mathcal{S}_k^*$ and the LII of $\mathcal{S}$, we just need to check this for geodesic domains which are symmetrical with respect to ∂S_k. Then, we must verify

$$A(G) \le c\, L(\partial_0 G)$$

for geodesic domains G of $\mathcal{S}_k$, such that $\partial G \cap \partial \mathcal{S}_k \neq \varnothing$, where $\partial_0 G$ means

$$\partial_0 G \equiv \partial G \setminus \partial \mathcal{S}_k .$$

Consider the open sets $C_t = \{p \in \mathcal{S}_k : d(p, \partial \mathcal{S}_k) < t\}$ for positive t. Let G_t be the geodesic domain "corresponding" to C_t (each puncture or boundary curve of G_t is freely homotopic to a boundary curve of C_t). If G_t is empty for all positive t, then $\mathcal{S}_k$ is a doubly connected domain (a funnel), $\mathcal{S}_k^*$ is an annulus, and the claim is true with constant 1. Then, we can assume that G_t is connected and not empty for $t \ge t_0$. G_t is non decreasing in t, and if $t_1 < t_2$ are such that $A(G_{t_1}) < A(G_{t_2})$, the constant curvature -1 and the Gauss-Bonnet theorem give $A(G_{t_1}) + 2\pi \le A(G_{t_2})$.

This implies that there exists a positive number T such that $G_t = G_T$ for all $t \ge T$, or $A(G_t) \to \infty$ as $t \to \infty$.

The first possibility is easy: there are only a finite number of geodesic domains. Without loss of generality, we can asssume that $A(G_t) \to \infty$ as $t \to \infty$.

Case 1. $A(G) \ge 2\, h(\mathcal{S})\, \ell$, with $\ell = \sum_{j=1}^{m} L(\gamma_j)$. In this case,

$$2\, h(\mathcal{S})\, \ell \le A(G) \le h(\mathcal{S})\, L(\partial G) \le h(\mathcal{S})\, \left(L(\partial_0 G) + \ell\right)$$

and

$$\ell \le L(\partial_0 G).$$

Therefore,

$$A(G) \le 2\,h(\mathcal{S})\,L(\partial_0 G).$$

Case 2. $A(G) < 2\,h(\mathcal{S})\,\ell$.

Let Ω be a geodesic domain in $\mathcal{S}_k$ such that

$$\partial\mathcal{S}_k \subset \partial\Omega \qquad \text{and} \qquad A(\Omega) \ge 2\,h(\mathcal{S})\,\ell.$$

We can choose Ω, for example, as the first geodesic domain G_t satisfying $A(G_t) \ge 2\,h(\mathcal{S})\,\ell$.

We define

$$a \equiv \min\left\{L(\gamma):\ \gamma \text{ closed simple geodesic, } \gamma \subset \overline{\Omega}\right\},$$
$$b \equiv \max\left\{L(\gamma):\ \gamma \text{ closed simple geodesic, } \gamma \subset \partial_0\Omega\right\}.$$

Since $A(\Omega) > A(G)$ and $\Omega \cap G \ne \varnothing$, one of the two next possibilities holds:

Case 2.1. There exists a closed simple geodesic $\gamma \subset \overline{\Omega} \cap \partial_0 G$. Then

$$L(\partial_0 G) \ge L(\gamma) \ge a.$$

Case 2.2. There exists a closed simple geodesic η in $\partial_0 G$, which meets some $\gamma \subset \partial_0\Omega$.

Then, the Collar Lemma [**R**] says that $L(\eta) \ge 4d_0$, where d_0 (the width of the greater collar of γ) satisfies

$$\cosh d_0 \ge \coth\frac{L(\gamma)}{2} \ge \coth\frac{b}{2},$$

and

$$d_0 \ge D \equiv \operatorname{arc\,cosh}\left(\coth\frac{b}{2}\right).$$

Randol [**R**] states the Collar Lemma if the surface is compact, but the same proof, without any change, works for a general Riemann surface.

Therefore,

$$L(\partial_0 G) \ge L(\eta) \ge 4\,D.$$

In both cases (2.1 and 2.2) $L(\partial_0 G) \ge \min\{a, 4D\} \equiv c_0$. Then

$$A(G) \le h(\mathcal{S})\,(L(\partial_0 G) + \ell) \le h(\mathcal{S})\,\left(L(\partial_0 G) + \ell\,\frac{L(\partial_0 G)}{c_0}\right)$$

and

$$A(G) \le h(\mathcal{S}) \left(1 + \frac{\ell}{c_0}\right) L(\partial_0 G).$$

Obviously, $\ell \ge a \ge c_0$ and $1 + \ell/c_0 \ge 2$. Therefore, in any case,

$$A(G) \le h(\mathcal{S}) \left(1 + \frac{\ell}{c_0}\right) L(\partial_0 G).$$

Consequently,

$$h(\mathcal{S}_k^*) \le 2 + h_g(\mathcal{S}_k^*) \le 2 + h(\mathcal{S}) \left(1 + \frac{\ell}{c_0}\right),$$

and the proof of Theorem 1 is now complete. ■

3. Proof of Theorem 2.

The desired Riemann surface $\mathcal{R}$ will be obtained with the help of a graph G. We will construct this graph in three steps.

In the set of vertices of any connected graph we can define a natural distance:

$$d(p,q) = \inf \{\text{length of the paths from } p \text{ to } q\}.$$

This will be "the distance" in all graphs of this section.

First, let T be the infinite complete binary tree with root r_0. Secondly, let V_n be the subset of 2^n vertices of T at distance n of r_0. We can construct new graphs G_n $(n \ge 1)$ with vertices V_n. In G_1 there is one edge between the two vertices of V_1. The edges of G_n $(n \ge 2)$ are chosen as follows: $2^n - 1$ vertices of V_n are connected by a complete binary tree with 2^{n-1} leaves and with root r_n (in any way); we add another edge between r_n and the last vertex v_n of V_n. In this way, the degree of the vertices of G_n is one (if the vertex is a leave) or three (if the vertex is not a leave). The leaves are at distance $n-1$ of r_n, except for v_n which is at distance 1. Hence, the diameter of G_n is $2n-2$, if $n \ge 2$.

Finally, we are ready to construct the graph G. The vertices of G are the vertices of T. The edges of G are the union of the edges of T and the edges of G_n, for all $n \ge 1$. The root r_0 of G has degree two. The other vertices of G have degree four or six.

To build up our Riemann surface $\mathcal{R}$, modelled upon the graph G, we will need the so called Löbell Y-pieces, which are a standard tool for constructing Riemann surfaces. A clear description of these Y-pieces and their use is given in [**C**, Chapter X.3].

A Löbell Y-piece is a three-holed sphere, endowed with a metric of constant negative curvature -1, so that the boundary curves are geodesics. We also require that the lengths of the boundary curves are the same, say 2α, and the distance between any two of these boundary curves is β, say. Then α and β are related by

$$\sinh\left(\frac{\alpha}{2}\right)\sinh\left(\frac{\beta}{2}\right)=\frac{1}{2}.$$

This is the unique restriction on α and β. See [**C**, p. 248] for details.

Fix α and β satisfying the above relation.

A X-piece (*-piece) is a four-holed (six-holed) sphere, endowed with a metric of curvature -1, so that the boundary curves are geodesics of the same length 2α. We can construct these pieces, for example, joining two (four) Y-pieces, by identifying corresponding boundary curves.

If we now put together these pieces following the combinatorial design of G, with the X-pieces (*-pieces) in the place of the vertices of degree four (six), we obtain a complete surface of constant negative curvature -1.

The only non-standard vertex is r_0, which has degree two. There is not problem if we forget r_0 and consider that the two vertices of V_1, are connected by a double edge.

Since we have used only two distinct pieces to build up $\mathcal{R}$, it is trivial to see that $\iota(\mathcal{R})>0$.

First of all, we will prove that $\mathcal{R}\in O_{HD}$. Let u be a harmonic function in $\mathcal{R}$ with finite Dirichlet integral. Without loss of generality we can assume that u is a bounded function [**AS**, p. 203] [**SN**, p. 178]. We want to verify that u is constant.

If u has limit at infinity, there is a point p in $\mathcal{R}$ such that $u(p)$ is the maximum or the minimum of u in $\mathcal{R}$. The maximum principle implies that u is constant.

If u is non-constant and has not limit at infinity, we can assume that u is positive and

$$\limsup_{z\to\infty} u(z)>4 \qquad \text{and} \qquad \liminf_{z\to\infty} u(z)<1.$$

The maximum (minimum) principle implies that each connected component of the set $\{u>4\}$ ($\{u<1\}$) is not a relatively compact set of $\mathcal{R}$.

This implies that, for each $n\geq n_0$, there exist points p_n, q_n in the pieces of $\mathcal{R}$ corresponding to G_n, such that

$$u(p_n)>4 \qquad \text{and} \qquad u(q_n)<1.$$

Since u is a positive harmonic function, the Harnack's Theorem says that there exists a positive number $\varepsilon < \iota(\mathcal{R})$, independent of n, such that,

$$u(z) \geq 3, \quad \text{for all} \quad z \in B(p_n, \varepsilon),$$
$$u(z) \leq 2, \quad \text{for all} \quad z \in B(q_n, \varepsilon).$$

Let the manifold with boundary $\mathcal{R}_n$ be the union of the pieces in $\mathcal{R}$ corresponding to the vertices V_n of G_n. We need a geodesic γ_n between p_n and q_n, completely contained in $\mathcal{R}_n$, which minimizes distance inside $\mathcal{R}_n$. To prove the existence of such geodesic, consider the Riemann surface Ω

$$\Omega \equiv \left\{ z \in \mathbb{C} : \ 1 < |z| < v^2 \right\},$$

where the constant v is chosen so that the geodesic $\{|z| = v\}$ has length 2α, the length of each boundary curve of $\mathcal{R}_n$. If we join a copy of $\Omega_0 \equiv \{1 < |z| \leq v\}$ in each boundary curve of $\mathcal{R}_n$, we obtain a new Riemann surface $\mathcal{R}_n^0$. Since $\mathcal{R}_n^0$ is complete, there is a geodesic γ_n between p_n and q_n such that the length L_n of γ_n is equal to the distance between p_n and q_n. γ_n is "completely contained in $\mathcal{R}_n$", because if γ enters in some copy of Ω_0, it lies there forever.

Consider now the Fermi coordinates (r, t) [**C**, p. 247], where $r \in [0, L_n]$ describes the curve γ_n, and $t \in [-\varepsilon, \varepsilon]$ describes the orthogonal geodesics to γ_n.

Observe that if we choose

$$\varepsilon < \frac{1}{2} \operatorname{arc\,cosh} \sqrt{\cosh \iota(\mathcal{R})} < \iota(\mathcal{R}),$$

$(0, L_n) \times (-\varepsilon, \varepsilon)$ corresponds injectively to a region $\Lambda_n \subset \mathcal{R}$. Let us denote by π this correspondence.

Assume that there exists two points $(r_1, t_1) \neq (r_2, t_2)$ in $(0, L_n) \times (-\varepsilon, \varepsilon)$, corresponding to the same $p \in \Lambda_n$. By the definition of $\iota(\mathcal{R})$, it is not possible that $(r_2, t_2) \in B((r_1, 0), \iota(\mathcal{R}))$. This implies $|r_1 - r_2| > 2\varepsilon$, because if $d \equiv d((r_2, t_2), (r_1, 0))$, hyperbolic trigonometry [**F**, p. 92] gives

$$\cosh(r_2 - r_1) \cosh t_2 = \cosh d \geq \cosh \iota(\mathcal{R}) > \cosh^2(2\varepsilon),$$

and we have

$$\cosh(r_2 - r_1) \cosh \varepsilon > \cosh(r_2 - r_1) \cosh t_2 > \cosh(2\varepsilon) \cosh \varepsilon.$$

Then

$$|r_2 - r_1| > 2\varepsilon,$$

and

$$d(\pi(r_1,0),\pi(r_2,0)) \le d(\pi(r_1,0),\pi(r_1,t_1)) + d(\pi(r_2,t_2),\pi(r_2,0)) = t_1 + t_2 < 2\varepsilon\,.$$

But $|r_1 - r_2| > 2\varepsilon$ and $d(\pi(r_1,0),\pi(r_2,0)) < 2\varepsilon$ contradict that γ_n minimizes length between p_n and q_n.

Therefore, $(0, L_n)\times(-\varepsilon,\varepsilon)$ corresponds injectively to a region $\Lambda_n \subset \mathcal{R}$.

It is easy to see that for all $t \in (-\varepsilon,\varepsilon)$, if $\gamma_n^t \equiv \{\pi(r,t) :\ 0 \le r \le L_n\}$,

$$1 \le u(\pi(0,t)) - u(\pi(L_n,t)) = \left| \int_{\gamma_n^t} \nabla u \, ds \right| \le \int_{\gamma_n^t} |\nabla u|\, ds \le \left(\int_{\gamma_n^t} |\nabla u|^2\, ds \right)^{1/2} \left(L(\gamma_n^t)\right)^{1/2}.$$

But the metric in Fermi coordinates is expressed by

$$ds^2 = \cosh^2 t\, dr^2 + dt^2,$$

and so

$$L(\gamma_n^t) = \int_0^{L_n} \cosh t\, dr = L_n \cosh t < L_n \cosh\varepsilon \le 2\, D\, n \cosh\varepsilon,$$

if D is the maximum of the diameters of the X-pieces and the *-pieces, because the diameter of G_n is $2n-2$. This gives

$$\int_{\gamma_n^t} |\nabla u|^2\, ds \ge \frac{1}{2Dn\cosh\varepsilon},$$

and

$$\int_{\Lambda_n} |\nabla u|^2 \ge \frac{\varepsilon}{Dn\cosh\varepsilon}.$$

Therefore

$$\int_{\mathcal{R}} |\nabla u|^2 \ge \frac{1}{2} \sum_{n\ge n_0} \frac{\varepsilon}{Dn\cosh\varepsilon} = \infty,$$

and so $u \notin HD(\mathcal{R})$.

This proves that $\mathcal{R} \in O_{HD}$. ■

To prove that $\mathcal{R}$ has a LII we need to precise the metric relationship between G and $\mathcal{R}$. Following Kanai's terminology [**K**], we say that an application φ, not necessarily continuous, between two metric spaces

$$\varphi : (M_1, d_1) \longrightarrow (M_2, d_2)$$

is a "rough isometry" if the following two conditions are satisfied:

(i) There are constants $a \geq 1$ and $b \geq 0$ such that

$$a^{-1} d_1(x,y) - b \leq d_2(\varphi(x),\varphi(y)) \leq a\, d_1(x,y) + b\,,$$

for all $x, y \in M_1$.

(ii) For some $\varepsilon > 0$, the ε-neighborhood of $\varphi(M_1)$ covers M_2.

A metric space M_1 is said to be "roughly isometric" to a metric space M_2 if there exists a rough isometry from M_1 into M_2. Obviously being roughly isometric is an equivalence relation between metric spaces.

It is evident that the graph G and the surface $\mathcal{R}$ are roughly isometric.

If F is a graph, for a subset P of vertices of F we define its "boundary" ∂P by

$$\partial P \equiv \{v \in V(F) : \; d(v,P) = 1\}\,.$$

If $|\cdot|$ denotes the cardinal of a subset of vertices, the "linear isoperimetric constant" of F is defined by

$$h(F) = \sup_P \frac{|P|}{|\partial P|},$$

where P ranges over all the non-empty finite subsets of vertices of F.

Combining two lemmas of Kanai [**K**, p. 401] one obtains that the surface $\mathcal{R}$ and the graph G verify a LII simultaneously. Moreover, the definition of the linear isoperimetric constant in a graph implies that G has a LII if the binary tree T has a LII, because both have the same vertices and G has more edges.

It is not difficult to prove, by induction in the number of vertices of P, that

$$|P| \leq |\partial P|,$$

for all non-empty finite subsets of vertices P of T. This complete the proof that $\mathcal{R}$ has a LII. ■

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