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On the Moser-Onofri and Prékopa-Leindler inequalities

Alessandro Ghigi

Dipartimento di Matematica "Felice Casorati"
Università degli Studi di Pavia, via Ferrata 1, 27100 Pavia, Italia
E-mail: ghigi@matapp.unimib.it

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Abstract

Using elementary convexity arguments involving the Legendre transformation and the Prékopa-Leindler inequality, we prove the sharp Moser-Onofri inequality, which says that

$$\frac{1}{16\pi}\int|\nabla\varphi|^2+\frac{1}{4\pi}\int\varphi-\log\left(\frac{1}{4\pi}\int e^{\varphi}\right)\geq 0$$

for any function $\varphi \in C^\infty(S^2)$.

Introduction

For an open bounded domain in $\mathbb{R}^n$, or more generally for an n-dimensional compact manifold M, the Sobolev embedding theorems say that $W^{1,p}(M)$ injects continuously in $L^{p^*}(M)$ for $1 < p < n$, and in $C^{1-n/p}(M)$ for $p > n$. In 1967 Trudinger proved that for the critical exponent $p = n$ there is a corresponding embedding in the Orlicz space of functions u such that $e^{u^{n/(n-1)}}$ is in L^q for some q (see e.g. [21, p. 25]). In [18] Moser computed the best value of q, both in the case where M is a bounded domain in $\mathbb{R}^n$ and when M is the two-dimensional sphere with its standard metric. In the latter case the optimal value of q is 4π:

Theorem 1. (Trudinger-Moser)

There is a constant $C > 0$ such that

$$\int_{S^2} e^{4\pi\varphi^2}\omega \leq C \tag{1}$$

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for any $\varphi \in W^{1,2}(S^2)$ *for which*

$$\int_{S^2} |\nabla\varphi|^2\omega \leq 1 \qquad \text{and} \qquad \int_{S^2} \varphi\omega = 0. \tag{2}$$

(Here ω denotes the volume form of the standard metric on S^2.) Moser used symmetrization to reduce the problem to a one-dimensional inequality, which is nevertheless quite non-trivial in both cases. Later A. Garsia and D. Adams [1] reproved Moser's result in the case of a bounded domain using Riesz potentials and O'Neill inequalities. Adams also obtained a much more general result allowing L^p bounds on higher order derivatives. The same approach was carried over to arbitrary compact manifolds by L. Fontana [12]. (See [11] for other related results.) Using the inequality

$$\varepsilon a^2 + \frac{b^2}{\varepsilon} \geq 2ab$$

one easily gets from Theorem 1 the following:

Theorem 2. (Moser)

There is a constant $C > 0$ *such that for all functions* $\varphi \in C^\infty(S^2)$

$$\frac{1}{4\pi}\int_{S^2} e^{\varphi}\omega \leq C\exp\left(\frac{1}{16\pi}\int_{S^2}|\nabla\varphi|^2\omega + \frac{1}{4\pi}\int_{S^2}\varphi\omega\right). \tag{3}$$

This is usually referred to as the *Moser-Trudinger inequality* and was Moser's original motivation for proving the optimal Sobolev embedding. Later Onofri [19] using conformal invariance and an estimate of Aubin [2] proved that one can indeed take $C = 1$ in (3) and characterized extremal functions:

Theorem 3. (Onofri)

For all $\varphi \in W^{1,2}(S^2)$

$$\frac{1}{4\pi}\int_{S^2} e^{\varphi}\omega \leq \exp\left(\frac{1}{16\pi}\int_{S^2}|\nabla\varphi|^2\omega + \frac{1}{4\pi}\int_{S^2}\varphi\omega\right). \tag{4}$$

Moreover equality is attained exactly for functions of the form

$$\psi(\xi) = -2\log(1 - \xi\cdot\zeta) + C \tag{5}$$

where C *is a constant,* $\xi \in S^2 \subset \mathbb{R}^3$ *and* ζ *is a fixed vector in* $\mathbb{R}^3$ *of norm less than 1.*

The importance of the Moser-Onofri inequality to geometry could hardly be overestimated and stems from a variety of roots: the Nirenberg problem of prescribing the Gaussian curvature of a conformal metric on S^2, extremals of regularised determinants, Kähler-Einstein metrics and Arakelov theory. We refer to [3, Chapter 8], [20], [22, Chapter 6], [15] and the beautiful survey by Sun-Yung Alice Chang [11] for an indication of these connections.

A number of other proofs of this inequality have been given in later years: [20] and [16] rely on Theorem 1, while [6] and [9] depend on a deep relation between Moser-Onofri inequality and the Hardy-Littlewood-Sobolev inequality. In dimension 2 a more direct proof can be given with the method of competing symmetries, see [10].

The purpose of this note is to present a new proof of the Moser-Onofri inequality (4) based on convexity arguments, especially the following well-known result in convex analysis, which is a functional version of the classical Brunn-Minkowski theorem.

Theorem 4. (Prékopa-Leindler)

Let f, g and m be nonnegative measurable functions on $\mathbb{R}^n$, such that

$$m\big(\lambda x + (1-\lambda)y\big) \geq f(x)^{\lambda} g(y)^{1-\lambda} \tag{6}$$

for all $x, y \in \mathbb{R}^n, \lambda \in [0,1]$. Then

$$\int_{\mathbb{R}^n} m \geq \left(\int_{\mathbb{R}^n} f\right)^{\lambda} \left(\int_{\mathbb{R}^n} g\right)^{1-\lambda}. \tag{7}$$

(For a simple proof see for example [5, Lecture 5] or [7].) Following Moser we reduce (3) to an inequality for a real function. Using the coarea formula, we show that this one-dimensional inequality follows from the lower boundedness of a functional Φ defined on convex functions z on the interval $(-1,1)$ and involving integrals of z and its Legendre transform (see Proposition 2). Convexity and boundedness of Φ easily follow from the Prékopa-Leindler inequality. This proves Moser's inequality (3). To get the best constant it is enough to show that the function $z = u_0^*$ (see (12)) corresponds to the minimum of Φ, and this follows from elementary convexity arguments.

Finally we give the characterization of the extremals in the spirit of [20, pp. 165ff]. Osgood, Phillips and Sarnak reasoned as follows: if ψ is an extremal of (4), then it satisfies the Euler-Lagrange equation for the corresponding functional (i.e. F in (8) below). This simply means that the conformal metric $e^{2\psi}g$ is of constant Gaussian curvature. Therefore this metric is isometric to the standard metric on the sphere, that is there is a diffeomorphism $f : S^2 \to S^2$ such that $f^*(e^{2\psi}g) = g$. But since $e^{2\psi}g$ and g are manifestly in the same conformal class, such a diffeomorphism must be a conformal transformation. This ensures that ψ is of the form (5).

Instead we use the Euler-Lagrange equation after symmetrizing the minimizer. In one dimension regularity issues simplify quite a lot, and we are able to give the characterization of extremals for general $W^{1,2}$ functions, taking advantage of an important result of Brothers and Ziemer on the extremals in the rearrangement inequality.

We recall that Onofri gave the characterization of the extremals, using a topological argument, showing that any smooth minimizer is related by a conformal transformation to another function satisfying both the Euler-Lagrange equation and the Kazdan-Warner conditions (i.e. eq. (8) in [19]). It is known that such a function must be constant.

We remark that we do not reprove the sharp Sobolev embedding, i.e. Theorem 1, but give a direct proof of Theorems 2 and 3. It would be interesting to find a proof of Theorem 1 along the lines of the present paper.

We should also remark that it does not seem possible to generalise the method used in this paper to higher dimensions.

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1. Proof of Moser inequality

For $\varphi \in C^\infty(S^2)$ put

$$F(\varphi) = \frac{1}{16\pi}\int_{S^2} |\nabla\varphi|^2\omega - \frac{1}{4\pi}\int_{S^2}\varphi\omega - \log\left(\frac{1}{4\pi}\int_{S^2} e^{-\varphi}\omega\right). \tag{8}$$

Moser inequality is of course equivalent to the fact that F is bounded below on $C^\infty(S^2)$, while Onofri inequality says it is nonnegative.

As a first step we apply symmetrization to reduce to a one-dimensional problem.

Lemma 1

Let $\mathcal{D}$ denote the space of functions on the sphere that are constant on parallel circles and that are constant near the poles. Then

$$\inf_{C^\infty(S^2)} F = \inf_{\mathcal{D}} F$$

Proof. Recall that *spherical symmetrization* is a process that to a smooth function φ on S^2 associates a function $\varphi^\#$, which is constant on the parallel circles, in such a way that

$$\int_{S^2} f(\varphi^\#) = \int_{S^2} f(\varphi) \quad \text{and} \quad \int_{S^2} |\nabla\varphi^\#|^2 \leq \int_{S^2} |\nabla\varphi|^2 \tag{9}$$

where f is any continuous function on the real line. (See e.g. [4, Corollary 3 p. 60] or [17].) One immediately checks that $F(\varphi^\#) \leq F(\varphi)$. A density argument based on the continuity of F in the $W^{1,2}$ norm shows that one can further reduce to $\mathcal{D}$. □

Denote by (θ, y) the usual coordinates on S^2, namely $\theta \in (-\pi/2,\ \pi/2)$ is the longitude, that is the signed distance from equator, and y is latitude, that we consider as a periodic (geodesic) parameter on the equator itself. Then the metric and the volume form are given by

$$g = d\theta^2 + \cos^2\theta\, dy^2 \qquad \omega = \cos\theta\, d\theta \wedge dy. \tag{10}$$

Put

$$x = \log\tan\left(\frac{\theta}{2} + \frac{\pi}{4}\right) \tag{11}$$

and use $(x, y) \in \mathbb{R}\times\mathbb{R}$ as coordinates on $S^2 \setminus \{\text{poles}\}$. ($z = x + \mathrm{i}\, y$ is in fact a complex parameter on $\mathbb{C}^* \subset \mathbb{P}^1(\mathbb{C}) = S^2$.) Put also

$$u_0(x) = \log\left(\frac{1+e^{2x}}{2e^x}\right) \tag{12}$$

and denote by V the space of smooth functions on the real line such that

$$\begin{cases} u = u_0 + a & \text{for } x \ll 0 \\ u = u_0 + b & \text{for } x \gg 0 \end{cases} \tag{13}$$

where a and b are constants depending on the function u. If $\varphi \in \mathcal{D}$ then it does not depend on y, and it is clear that

$$u(x) = u_0(x) + \frac{\varphi(x)}{2} \tag{14}$$

belongs to V. The next step is to give an expression of $F(\varphi)$ in terms of u. For $\varphi \in C^\infty(S^2)$ put

$$B(\varphi) = \frac{1}{16\pi}\int_{S^2} |\nabla\varphi|^2\omega - \frac{1}{4\pi}\int_{S^2}\varphi\omega \qquad A(\varphi) = \log\left(\frac{1}{4\pi}\int_{S^2} e^{-\varphi}\omega\right).$$

Clearly $F = B - A$. Finally for $u \in V$ put

$$E(u) = \int_{-\infty}^{+\infty} \big(xu'(x) - u(x)\big)u''(x)\,dx.$$

Proposition 1

E is a well-defined functional on V. Moreover for $\varphi \in \mathcal{D}$ and u as in (14)

$$B(\varphi) = E(u) - E(u_0) \tag{15}$$

$$A(\varphi) = \log\left(\frac{1}{2}\int_{-\infty}^{+\infty} e^{-2u(x)}\,dx\right). \tag{16}$$

Proof. To prove that E is well-defined it is enough to show that for $u \in V$, $u'' \in L^1$ and $xu' - u \in L^\infty$. Since

$$u_0' = \frac{e^{2x}-1}{e^{2x}+1} \qquad u_0'' = \frac{4e^{2x}}{(e^{2x}+1)^2} \tag{17}$$

the case $u = u_0$ follows from direct computation. To extend to general $u \in V$ it is enough to consider (13). Next fix $\varphi \in \mathcal{D}$ and compute

$$\theta = 2\arctan e^x - \frac{\pi}{2} \qquad \nabla\varphi = \frac{\partial\varphi}{\partial\theta}\frac{\partial}{\partial\theta} = \varphi'\frac{dx}{d\theta}\frac{\partial}{\partial\theta}$$

$$\theta' = \frac{2e^x}{1+e^{2x}} \qquad |\nabla\varphi|^2 = \frac{(\varphi')^2}{(\theta')^2} = \frac{(\varphi')^2}{u_0''}$$

$$\left|\frac{\partial}{\partial\theta}\right|^2 = 1 \qquad \nabla\varphi|^2\omega = (\varphi')^2\,dx\wedge dy$$

$$\frac{1}{16\pi}\int_{S^2}|\nabla\varphi|^2\omega = \frac{1}{8}\int_{-\infty}^{+\infty}(\varphi')^2.$$

Since φ' has compact support we can integrate by parts:

$$\frac{1}{16\pi}\int_{S^2}|\nabla\varphi|^2\omega = \frac{1}{8}\int_{-\infty}^{+\infty}(\varphi')^2 = -\frac{1}{8}\int_{-\infty}^{+\infty}\varphi\varphi''$$

$$= -\frac{1}{2}\int_{-\infty}^{+\infty} uu'' + \frac{1}{2}\int_{-\infty}^{+\infty} uu_0'' + \frac{1}{2}\int_{-\infty}^{+\infty} u_0u'' - \frac{1}{2}\int_{-\infty}^{+\infty} u_0u_0''.$$

On the other hand

$$\frac{1}{4\pi}\int_{S^2}\varphi\omega = \frac{1}{2}\int_{-\infty}^{+\infty}\varphi u_0'' = \int_{-\infty}^{+\infty} u u_0'' - \int_{-\infty}^{+\infty} u_0 u_0''$$

$$B(\varphi) = -\frac{1}{2}\int_{-\infty}^{+\infty} u u'' + \frac{1}{2}\int_{-\infty}^{+\infty} u_0 u_0'' + \frac{1}{2}\int_{-\infty}^{+\infty}(u_0 u'' - u u_0'').$$

The last integral contains some asymptotic information: for $R \gg 0$ integration by parts gives

$$\begin{aligned}\int_{-R}^{R}(u_0 u'' - u u_0'') &= \Big[u_0 u' - u u_0'\Big]_{-R}^{R}\\ &= u_0(R)u_0'(R) - \big(u_0(R)+b\big)u_0'(R)\\ &\quad + \ -u_0(-R)u_0'(-R) + \big(u_0(-R)+a\big)u_0'(-R).\end{aligned}$$

Letting R tend to ∞ we get

$$\int_{-\infty}^{+\infty}(u_0 u'' - u u_0'') = -(a+b)$$

whence

$$B(\varphi) = -\frac{1}{2}\int_{-\infty}^{+\infty} u u'' + \frac{1}{2}\int_{-\infty}^{+\infty} u_0 u_0'' - \frac{1}{2}(a+b). \tag{18}$$

(Here a and b are as in (13) so they depend on u.) On the other hand

$$\int_{-R}^{R} x u' u'' = \Big[(xu')u'\Big]_{-R}^{R} - \int_{-\infty}^{+\infty}(u' + xu'')u'$$

$$\begin{aligned}\int_{-R}^{R} x u' u'' &= \frac{1}{2}\Big[x(u')^2\Big]_{-R}^{R} - \frac{1}{2}\int_{-\infty}^{+\infty}(u')^2\\ &= \frac{1}{2}\Big[x(u')^2\Big]_{-R}^{R} - \frac{1}{2}\Big[uu'\Big]_{-R}^{R} + \frac{1}{2}\int_{-R}^{R} u u''.\end{aligned}$$

If $R \gg 0$

$$\begin{aligned}&\int_{-R}^{R} x u' u'' - \int_{-R}^{R} x u_0' u_0''\\ &\qquad = \frac{1}{2}\int_{-R}^{R} u u'' - \frac{1}{2}\int_{-R}^{R} u_0 u_0'' - \frac{1}{2}\Big[uu' - u_0 u_0'\Big]_{-R}^{R}\end{aligned}$$

and again

$$\Big[uu' - u_0 u_0'\Big]_{-R}^{R} = -(a+b),$$

so

$$\int_{-\infty}^{+\infty} x u' u'' - \int_{-\infty}^{+\infty} x u_0 u_0'' = \frac{1}{2}\int_{-\infty}^{+\infty} u u'' - \frac{1}{2}\int_{-\infty}^{+\infty} u_0 u_0'' + \frac{1}{2}(a+b)$$

and

$$E(u) - E(u_0) = \int_{-\infty}^{+\infty} xu'u'' - \int_{-\infty}^{+\infty} xu_0u_0'' - \int_{-\infty}^{+\infty} uu'' + \int_{-\infty}^{+\infty} u_0u_0''$$
$$= -\frac{1}{2}\int_{-\infty}^{+\infty} uu'' + \frac{1}{2}\int_{-\infty}^{+\infty} u_0u_0'' + \frac{1}{2}(a+b) = B(\varphi).$$

This proves (15). To prove (16) observe that $u_0'' = e^{-2u_0}$. (This expresses the fact that the metric on S^2 is Kähler-Einstein.) Therefore

$$\frac{1}{4\pi}\int_{S^2} e^{-\varphi}\omega = \frac{1}{4\pi}\int_{S^2} e^{-\varphi}e^{-2u_0}\,dx\wedge dy = \frac{1}{2}\int_{-\infty}^{+\infty} e^{-2u(x)}\,dx$$

which proves (16). □

For a function $u : \mathbb{R}^n \to (-\infty, \infty]$ denote by u^* its Legendre transform, which is defined by the formula

$$u^*(y) = \sup_{x\in\mathbb{R}}\big(xy - u(x)\big). \tag{19}$$

If u is defined only on a subset $\Omega \subset \mathbb{R}^n$ one first extends it to all $\mathbb{R}^n$ by putting it equal to $+\infty$ on $\mathbb{R}^n \setminus \Omega$, then applies the formula above to define its Legendre transform.

We will prove that F is bounded below by a functional depending on u^* only.

Lemma 2

a) *If u_1 and u_2 are functions on $\mathbb{R}^n$, then $||u_1^* - u_2^*||_\infty \leq ||u_1 - u_2||_\infty$.*
b) $u_0^*(y) = \frac{1}{2}(1+y)\log(1+y) + \frac{1}{2}(1-y)\log(1-y)$ *which is a continuous function on* $[-1,1]$.
c) *If $u \in V$, then $u^* \in L^\infty(-1,1)$.*
d) *If $u \in V$ and $y \in (-1,1)$, the supremum in* (19) *is attained at some point x such that $u'(x) = y$; therefore for $y \in (-1,1)$*

$$u^*(y) = \max_{u'(x)=y}\big(xy - u(x)\big). \tag{20}$$

Proof. (a) From the definition (19)

$$u_1^*(y) = \sup_{x\in\mathbb{R}^n}\big(xy - u_1(x)\big)$$
$$\leq \sup_{x\in\mathbb{R}^n}\big(xy - u_2(x)\big) + \sup_{x\in\mathbb{R}^n}\big(u_2(x) - u_1(x)\big)$$
$$\leq u_2^*(y) + ||u_1 - u_2||_\infty.$$

Interchanging u_1 and u_2 and taking the sup in y one gets the result. Note that this still holds when the functions attain infinite values. Indeed, if the set where they are infinite is not the same for both, clearly $||u_1 - u_2||_\infty = \infty$ and there is nothing to prove. While if they are both finite on the same set $\Omega \subset \mathbb{R}^n$ it is obviously enough to compute the suprema above on the set Ω. (We use the convention $\infty - \infty = 0$.) (b) is an elementary computation. (c) follows from (a) and (b). To prove (d), let $|y| < 1$. Then $xy - u(x) = x(y-1) + \big(x - u(x)\big)$. Now $x - u(x) = x - u_0(x) + \big(u_0(x) - u(x)\big)$ is bounded for $x > 0$. On the other hand as $x \to +\infty$, $x(y-1)$ tends to $-\infty$. Therefore $\lim_{x\to+\infty}\big(xy - u(x)\big) = -\infty$ and similarly for $x \to -\infty$. Hence the supremum is attained at some point $\bar{x}$. But $z(x) = yx - u(x)$ is a smooth function of x, therefore $z'(\bar{x}) = y - u'(\bar{x}) = 0$. □

Lemma 3

If $u \in V$ then

$$E(u) \geq \int_{-1}^{1} u^*(y)\,dy. \tag{21}$$

Moreover the equality holds if u is strictly convex.

Proof. As noted in the proof of Proposition 1, $w(x) = xu'(x) - u(x) \in L^\infty$. Assume $w \geq -M$ for some $M \in \mathbb{R}$, and put $\bar{u}(x) = u(x) - M$. Then $\bar{u}' = u'$ and

$$\bar{w}(x) = x\bar{u}'(x) - \bar{u}(x) = w + M \geq 0.$$

Moreover $\bar{u}^* = u^* - M$. Since

$$\lim_{x\to\pm\infty} u'(x) = \lim_{x\to\pm\infty} u_0'(x) = \pm 1, \int u'' = 2,$$

$E(u) = E(\bar{u}) - 2M$ and $\int_{-1}^{1} \bar{u}^* = \int_{-1}^{1} u^* - 2M$. Therefore it is enough to prove (21) for $u = \bar{u}$. Put $f = \bar{u}' : \mathbb{R} \to \mathbb{R}$. It follows from the coarea formula [13, p. 82, Theorem 2] that

$$E(\bar{u}) = \int_{-\infty}^{+\infty} \bar{w}(x) f'(x)\,dx = \int_{-\infty}^{+\infty} \Big[\sum_{f^{-1}(y)} \bar{w}(x) \Big]\,dy.$$

Since $\bar{w} \geq 0$

$$E(\bar{u}) \geq \int_{-1}^{1} \Big[\sum_{f^{-1}(y)} \bar{w}(x) \Big]\,dy.$$

But again from $\bar{w} \geq 0$ and (20) it follows that $\sum_{f^{-1}(y)} \bar{w}(x) \geq \bar{u}^*(y)$, whence the result. Finally, if u is strictly convex $u^*(u'(x)) = xu'(x) - u(x) = w(x)$, and it is enough to make the substitution $y = u'(x)$ to prove the equality in (21). □

Let W denote the space of bounded convex functions on $(-1, 1)$. For $z \in W$ put

$$\Phi(z) = \int_{-1}^{1} z(y)\,dy - \log\Big(\frac{1}{2} \int_{-\infty}^{+\infty} e^{-2z^*(x)}\,dx \Big). \tag{22}$$

Proposition 2

The functional Φ is well-defined and finite on W. If $\varphi \in \mathcal{D}$ and $u = u_0 + \varphi/2$, then $u^* \in W$ and

$$F(\varphi) \geq \Phi(u^*) - E(u_0) \tag{23}$$

Moreover equality holds if u is strictly convex. In particular $\inf_{\mathcal{D}} F \geq \inf_W \Phi - E(u_0)$.

Proof. If $z \in W$, then $||z - u_0^*||_\infty < \infty$ since both z and u_0^* are bounded. Thanks to Lemma 2 (a) $||z^* - u_0||_\infty < \infty$ as well, therefore the integral inside the logarithm in (22) converges and $\Phi(z)$ is well-defined for $z \in W$. If $\varphi \in \mathcal{D}$, then $u \in V$ and u^* is bounded by Lemma 2 (c), so $u^* \in W$. Using (16) and recalling that $u^{**} \le u$ for any real function, it is immediate that

$$A(\varphi) \le \log\Big(\frac{1}{2}\int_{-\infty}^{+\infty} e^{-2u^{**}}\Big). \tag{24}$$

Together with (15) and (21) this yields (23). □

We now introduce the Prékopa-Leindler inequality in the following form.

Lemma 4

Let φ, ψ and μ be nonnegative measurable functions on $[0, \infty)$, such that

$$\mu\big(x^\lambda y^{1-\lambda}\big) \ge \varphi(x)^\lambda \psi(y)^{1-\lambda} \tag{25}$$

for all $x, y \in [0, \infty), \lambda \in [0, 1]$. Then

$$\int_0^\infty \mu \ge \Big(\int_0^\infty \varphi\Big)^\lambda \Big(\int_0^\infty \psi\Big)^{1-\lambda}. \tag{26}$$

Proof. Put $f(x) = \varphi(e^x)e^x$, $g(x) = \psi(e^x)e^x$ and $m(x) = \mu(e^x)e^x$. Then f, g, m satisfy (6). To get the result it is enough to apply Theorem 4 and to use the formula for the change of variables in order to check that the integrals in (7) coincide with the ones in (26). □

Lemma 5

Let z_1, z_2 be functions on a convex subset $\Omega \subset \mathbb{R}^n$. For $\lambda \in [0, 1]$ put $z = \lambda z_1 + (1-\lambda) z_2$. Denote by z_1^*, z_2^*, z^* the Legendre transforms of z_1, z_2 and z respectively. Then for any $x, y \in \Omega$

$$z^*\big(\lambda x + (1-\lambda)y\big) \le \lambda z_1^*(x) + (1-\lambda) z_2^*(y). \tag{27}$$

Proof. It is enough to apply (19):

$$\begin{aligned}
&\lambda z_1^*(x) + (1-\lambda) z_2^*(y) \\
&\quad = \lambda \sup_{\xi\in\Omega}\big\{x\cdot\xi - z_1(\xi)\big\} + (1-\lambda)\sup_{\eta\in\Omega}\big\{y\cdot\eta - z_2(\eta)\big\} \\
&\quad \ge \sup_{\xi\in\Omega}\big\{\lambda\big(x\cdot\xi - z_1(\xi)\big) + (1-\lambda)\big(y\cdot\xi - z_2(\xi)\big)\big\} \\
&\quad = \sup_{\xi\in\Omega}\big\{\big(\lambda x + (1-\lambda)y\big)\cdot\xi - z(\xi)\big\} \\
&\quad = z^*\big(\lambda x + (1-\lambda)y\big). \ \square
\end{aligned}$$

Theorem 5

The functional $\Phi : W \to \mathbb{R}$ is convex and bounded below.

Proof. Note first that W is a convex subset of $C^0(-1,1)$, so it makes sense to talk about convexity of functional Φ. The first term of Φ in (22) is linear so convex. It is enough to check that the second is concave. Let $z_1, z_2 \in W$, $\lambda \in [0,1]$ and put $z = \lambda z_1 + (1-\lambda) z_2$. It follows from Lemma 5 that

$$\begin{aligned} e^{-2z^*(\lambda x + (1-\lambda) y)} &\geq e^{-2\lambda z_1^*(x) - 2(1-\lambda) z_2^*(y)} \\ &= \big(e^{-2z_1^*(x)}\big)^\lambda \big(e^{-2z_2^*(y)}\big)^{(1-\lambda)}. \end{aligned}$$

Applying Lemma 4 we get

$$\begin{aligned} &\left(\int_{-\infty}^{\infty} e^{-2z^*}\right) \geq \left(\int_{-\infty}^{\infty} e^{-2z_1^*}\right)^\lambda \left(\int_{-\infty}^{\infty} e^{-2z_2^*}\right)^{(1-\lambda)} \\ &\qquad \log\left(\frac{1}{2}\int_{-\infty}^{\infty} e^{-2z^*(x)}\,dx\right) \\ &\qquad \geq \lambda \log\left(\frac{1}{2}\int_{-\infty}^{\infty} e^{-2z_1^*(x)}\,dx\right) + (1-\lambda)\log\left(\frac{1}{2}\int_{-\infty}^{\infty} e^{-2z_2^*(x)}\,dx\right). \end{aligned}$$

Therefore the second term in (22) is concave and Φ is convex. If $z \in W$ put $w(y) = z(-y)$. Clearly $w \in W$ and $w^*(x) = z^*(-x)$, hence $\Phi(w) = \Phi(z)$. From the convexity of Φ it follows that

$$\Phi(\bar{z}) \leq \frac{\Phi(z) + \Phi(w)}{2} = \Phi(z)$$

where $\bar{z} = (z+w)/2$. To compute the infimum of Φ we can therefore restrict to *even* functions. For such a function $z \in W$

$$\Phi(z) = 2\int_0^1 z(y)\,dy - \log\left(\int_0^\infty e^{-2z^*(x)}\,dx\right).$$

Using Jensen inequality

$$\begin{aligned} e^{-\Phi(z)} &= \exp\left(-2\int_0^1 z(y)\,dy\right)\int_0^\infty e^{-2z^*(x)}\,dx \\ &\leq \int_0^1 e^{-2z(y)}\,dy \int_0^\infty e^{-2z^*(x)}\,dx. \end{aligned} \tag{28}$$

So it is enough to show that for some constant C and for any even function $z \in W$ we have

$$\int_0^1 e^{-2z(y)}\,dy \int_0^\infty e^{-2z^*(x)}\,dx \leq C. \tag{29}$$

Put

$$\begin{aligned} \psi(x) &= e^{-2z^*(x)} \\ \mu(t) &= e^{-t^2} \end{aligned} \qquad \varphi(y) = \begin{cases} e^{-2z(y)} & y \in [0,1] \\ 0 & y \in (1,\infty). \end{cases}$$

The fundamental property of the Legendre transformation, namely that

$$z(y) + z^*(x) \geq xy,$$

implies that

$$\sqrt{\varphi(y)\psi(x)} \leq \mu(\sqrt{xy})$$

i.e. (25) with $\lambda = 1/2$. Using Lemma 4 (i.e. the Prékopa-Leindler inequality) we conclude that

$$\sqrt{\Big(\int_0^\infty f\Big)\Big(\int_0^\infty g\Big)} \le \int_0^\infty e^{-t^2}\,dt = \frac{\sqrt{\pi}}{2}.$$

Taking the square we get (29) with $C = \pi/4$. This concludes the proof of the theorem. □

Proof of Theorem 2. It is enough to piece together Lemma 1, Proposition 2 and Theorem 5. □

2. Proof of Onofri inequality

Recall the following well-known property of convex functionals. We stress that it does not need *any* topological assumption. The proof is elementary and is left to the reader.

Lemma 6

Let L be a real vector space (of arbitrary dimension), $C \subset L$ a convex subset and $\Psi : C \to \mathbb{R}$ a convex functional. Let $x \in C$ and assume that for any $y \in C$ the directional derivative of Ψ at x in the direction $v = y - x$ exists and vanishes:

$$\frac{\mathrm{d}}{\mathrm{dt}}\Big|_{t=0} \Psi(x + tv) = 0. \tag{30}$$

Then Ψ attains its minimum at x.

The following lemma computes the differential of the Legendre transform as a nonlinear map between manifolds of convex functions.

Lemma 7

Let z_t (for $|t| < \varepsilon$) be a path of functions on $(a,b) \subset \mathbb{R}$, such that $z(t,y) = z_t(y)$ be a smooth function on $(-\varepsilon,\varepsilon) \times (a,b)$. Assume that each z_t is strictly convex in y. Let z_t^* be the path of their Legendre transforms, and put $z^*(t,x) = z_t^*(x)$. Then

$$\frac{\partial z^*}{\partial t}(t,x) = -\frac{\partial z}{\partial t}\Big(t, \frac{\partial z^*}{\partial x}(t,x)\Big). \tag{31}$$

Proof. Since z_t is strictly convex

$$z^*(t,x) + z^*\Big(t, \frac{\partial z^*}{\partial x}(t,x)\Big) = x\frac{\partial z^*}{\partial x}(t,x).$$

Differentiating with respect to t

$$\begin{aligned}\frac{\partial z^*}{\partial t}(t,x) + \frac{\partial z}{\partial t}\Big(t, \frac{\partial z^*}{\partial x}(t,x)\Big) + \frac{\partial z}{\partial y}\Big(t, \frac{\partial z^*}{\partial x}(t,x)\Big)\frac{\partial^2 z^*}{\partial x^2}(t,x)&\\ = x\frac{\partial^2 z^*}{\partial x^2}(t,x).&\end{aligned}$$

Since

$$\frac{\partial z}{\partial y}\Big(t, \frac{\partial z^*}{\partial x}(t,x)\Big) = x$$

we get (31). □

Proof of Theorem 3. We apply Lemma 6 with $L = L^\infty(-1,1)$, $C = W$, $\Psi = \Phi$ and $x = u_0^*$. It is enough to show that

$$\frac{\mathrm{d}}{\mathrm{dt}}\Big|_{t=0} \Phi(u_0^* + tv) = 0 \tag{32}$$

whenever $v = u_0^* - z_1$ and $z_1 \in W$. Put $z_0 = u_0^*$ and $z_t = u_0^* + tv = tz_1 + (1-t)z_0$. Since z_1 is convex and z_0 is *strictly* convex, then z_t is strictly convex as well. Differentiating under the integral sign

$$\frac{\mathrm{d}}{\mathrm{dt}}\Big|_{t=0} \Phi(z_t) = \int_{-1}^{1} v(y)\,dy + \int_{-\infty}^{+\infty} \frac{\partial z^*}{\partial t}(0,x) u_0''(x)\,dx$$

since $\int e^{-2u_0} = \int u_0'' = 2$. Using (31)

$$\begin{aligned}\int_{-\infty}^{+\infty} \frac{\partial z^*}{\partial t}(0,x) u_0''(x)\,dx &= -\int_{-\infty}^{+\infty} \frac{\partial z^*}{\partial t}\Big(0, \frac{\partial z^*}{\partial x}(0,x)\Big) u_0''\,dx \\ &= -\int_{-\infty}^{+\infty} v\big(u_0'(x)\big) u_0''(x)\,dx = -\int_{-1}^{1} v(y)\,dy.\end{aligned}$$

Therefore we can apply Lemma 6 to the effect that $\inf_W \Phi = \Phi(u_0^*)$. Lemma 1 and Proposition 2 yield then

$$\inf_{C^\infty(S^2)} F \geq \Phi(u_0^*) - E(u_0).$$

Since u_0 is strictly convex Lemma 3 implies $E(u_0) = \int_{-1}^{1} u_0^*$. Moreover $\int e^{-2u_0} = \int u_0'' = 2$, therefore $\Phi(u_0^*) = E(u_0)$ and $F \geq 0$. This completes the proof of the Moser-Onofri inequality (4).

Next we give a short argument to deal with the equality case. It uses neither Legendre transformation nor the Prékopa-Leindler inequality. On the other hand it relies on a deep result of Brothers and Ziemer on the extremals of rearrangement inequalities.

Let $\psi \in W^{1,2}(S^2)$ be an extremal of (4), i.e. $F(-\psi) = 0$. Denote by $\varphi = (-\psi)^\#$ the spherical symmetrization of $-\psi$. It belongs to the Sobolev space $W^{1,2}(S^2)$ too. It follows from properties (9) that

$$\int_{S^2} |\nabla\varphi|^2 = \int_{S^2} |\nabla\psi|^2, \tag{33}$$

and that $F(\varphi) = F(-\psi) = 0$. This means that φ is a minimiser of F, hence a weak solution of the Euler-Lagrange equation

$$\Delta\varphi + 4 - 4\frac{e^{-\varphi}}{\frac{1}{4\pi}\int_{S^2} e^{-\varphi}\omega} = 0. \tag{34}$$

If

$$c = \log\Big(\frac{1}{4\pi}\int_{S^2} e^{-\varphi}\omega\Big) \tag{35}$$

then

$$u = u_0 + \frac{1}{2}(\varphi + c) \in W^{1,2}_{\mathrm{loc}}(\mathbb{R})$$

is a weak solution of the equation $u'' = e^{-2u}$ on the real line. Since $W^{1,2}_{\mathrm{loc}}(\mathbb{R}) \subset C^0(\mathbb{R})$, $u \in C^0(\mathbb{R})$ and $e^{-2u} \in C^0(\mathbb{R}) \subset L^2_{\mathrm{loc}}(\mathbb{R})$. Standard regularity theory (see e.g. [14, Theorem 8.8 p. 183]) ensures then that $u \in W^{2,2}_{\mathrm{loc}}(\mathbb{R}) \subset C^1(\mathbb{R})$. So $e^{-2u} \in C^1 \subset W^{1,2}_{\mathrm{loc}}(\mathbb{R})$. Therefore u belongs to $W^{3,2} \subset C^2$ and is a classical solution of the ordinary differential equation $u'' = e^{-2u}$ defined on the whole real line. Since $u'' > 0$, $u'(x)$ is increasing and it has a limit for $x \to \pm\infty$. Since $u_0'(x) \to \pm 1$ as $x \to \pm\infty$, $\varphi'(x) = 2u'(x) - 2u_0'(x)$ has limits as well. In order for φ' to be in $L^2(\mathbb{R})$ these limits must vanish. Since

$$\frac{1}{8}\int \big(\varphi'(x)\big)^2\,dx = \frac{1}{16\pi}\int_{S^2} |\nabla\varphi|^2\omega < \infty,$$

φ' is indeed square-integrable, and we deduce

$$\lim_{x\to\pm\infty} \varphi'(x) = 0 \qquad \lim_{x\to\pm\infty} u'(x) = \lim_{x\to\pm\infty} u_0'(x) = \pm 1.$$

Let x_0 be the (unique) point such that $u'(x_0) = 0$. Put $a = e^{-u(x_0)}$. Then $u(x) = u_0(a(x - x_0)) - \log a$. Indeed this function is a solution of the equation with the same initial conditions at $x = x_0$ as u. By the definition of c, (35),

$$\int_{S^2} e^{-(\varphi+c)}\omega = 4\pi, \text{ i.e. } \int_{-\infty}^{\infty} e^{-2u} = \int_{-\infty}^{\infty} e^{-2u_0} = 2.$$

Therefore $a = 1$, i.e. $u(x) = u_0(x - x_0)$ is simply a translation of u_0 and

$$\varphi(x) = 2u_0(x - x_0) - 2u(x) - c.$$

Put

$$A = \frac{1 + e^{2x_0}}{2e^{x_0}} \qquad \varepsilon = \frac{e^{2x_0} - 1}{e^{2x_0} + 1},$$

then

$$\varphi(x) = 2\log\bigg(1 - \varepsilon\frac{e^{2x} - 1}{e^{2x} + 1}\bigg) + 2\log A - c.$$

Denote by ξ a point on $S^2 \subset \mathbb{R}^3$. Using polar coordinates (θ, y) as above, $\xi = (\cos\theta\cos y, \cos\theta\sin y, \sin\theta)$ and

$$\frac{e^{2x} - 1}{e^{2x} + 1} = \sin\theta$$

(see (11)). Therefore,

$$\varphi(\xi) = 2\log\big(1 - \xi\cdot(0,0,\varepsilon)\big) + c - \log A.$$

Observe that the only critical points of this function are the North and the South pole, which are respectively a maximum and a minimum point. Moreover $-\psi$ and $\varphi = (-\psi)^{\#}$ have the same Dirichlet integral, see (33). Therefore we can apply Theorem 5.1 in [8] to conclude that there is a transformation $R \in \mathrm{O}(3)$ such that $-\psi = \varphi \circ R$. This means that

$$\psi(\xi) = -2\log(1 - \xi\cdot\zeta) + C$$

with $C = \log A - c$ and $\zeta = R^{-1}(0, 0, \varepsilon) \in \mathbb{B}^3$. This proves that the extremals of the inequality have the desired form and completes the proof of Onofri's theorem. □

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