SOME PROPERTIES OF POLYNOMIAL SOLUTIONS OF A CLASS OF RICCATI-TYPE DIFFERENTIAL EQUATIONS

MIRA BHARGAVA

and

H. KAUFMAN

The equation

$$A(x)\,y' = B_0(x) + B_1(x)\,y + \dots + B_n(x)\,y^n \quad \left(' = \frac{d}{dx}\right) \quad (1)$$

comprises a variety of special types of first order equations such as the linear equation, Riccati's equation, Bernoulli's equation, and Abel's equation of the first kind. Assuming that A, B_0, $B_1, \dots, B_n$ are polynomials in x, the problem of the possible degrees of polynomial solutions of (1) was completely settled in [1]; the results include those of CAMPBELL and GOLOMB for the Riccati equation [2].

The nature of the polynomial solutions of (1) with $B_n(x) = 1$ was investigated in [3], the main results being embodied in Theorem 2 of that paper. The equation considered in [3] is

$$A(x)\,y' = B_0(x) + B_1(x)\,y + \dots + B_{n-2}(x)\,y^{n-2} + y^n \quad (2)$$

It should be noted that replacing y by $y - (B_{n-1}/n)$ in (1) (with $B_n = 1$) reduces (1) to the form (2). One of the results of Theorem 2 [3] is that subject to certain conditions on the degrees of the polynomials in equation (2), $\omega_i S$ is a solution of (2) if and only if

$$A\,\omega_i\,S' = -Q + \sum_{j=1}^{n-2} B_j\,\omega_i^j\,S^j$$

where S denotes the polynomial part of the expansion of $(-B_0)^{1/n}$ in descending powers of x, Q a polynomial such that $-B_0 = S^n + Q$, and ω_i the n^{th} roots of unity ($i = 1, \dots, n$).

The purpose of the present paper is to establish the form that (2) must take in order that all the $\omega_i S$ be solutions, and to show that for a certain form of (2) there is at most one possible solution. We first note that the n^{th} roots of unity, ω_i, can be represented in terms of a primitive n^{th} root ω as $\omega^0, \omega, \omega^2, \ldots, \omega^{n-1}$.

THEOREM 1. The polynomials $\omega^i S$ $(i = 0, 1, \ldots, n-1)$ are all solutions of (2) if and only if (2) has the form

$$A(x)\, y' = B_0(x) + B_1(x)\, y + y^n \tag{3}$$

where $B_0 = -S^n$ and S is a polynomial solution of $A(x)\, y' = B_1(x)\, y$.

PROOF: First let $\omega^i S$ be solutions of equation (2). Then by substituting $y = \omega^i S$ in equation (2), we get

$$A\omega^i S' = -Q + B_1 \omega^i S + B_2 \omega^{2i} S^2 + \ldots + B_{n-2}\, \omega^{(n-2)i}\, S^{n-2}$$

$$(i = 0, 1, \ldots, n-1).$$

Multiplying the above equations by ω^{-ki} $(i = 0, 1, \ldots, n-1)$ respectively and adding we have

$$A \sum_{i=0}^{n-1} \omega^{(1-k)i} S' = -Q \sum_{i=0}^{n-1} \omega^{-ki} + B_1 \sum_{i=0}^{n-1} \omega^{(1-k)i} S$$

$$+ B_2 \sum_{i=0}^{n-1} \omega^{(2-k)i} S^2 + \ldots + B_{n-2} \sum_{i=0}^{n-1} \omega^{(n-2-k)i} S^{n-2}$$

Setting $k = 0$ we obtain $Q = 0$ whence $B_0 = -S^n$.

The above equation then becomes:

$$A \sum_{i=0}^{n-1} \omega^{(1-k)i} S' = B_1 \sum_{i=0}^{n-1} \omega^{(1-k)i} S + B_2 \sum_{i=0}^{n-1} \omega^{(2-k)i} S^2$$

$$+ \ldots + B_{n-2} \sum_{i=0}^{n-1} \omega^{(n-2-k)i} S^{n-2}$$

With $k = 1$ we have $AS' = B_1 S$.

Hence S is a polynomial solution of the equation

$$A\, y' = B_1\, y.$$

Similarly, by setting $k = 2, 3, \ldots, n-2$ in turn we find that $B_i = 0$ $(i = 2, \ldots, n-2)$. Equation (2) thus reduces to the form

$$A\,y' = B_0 + B_1\,y + y^n,$$

where $B_0 = -S^n$ and S is a polynomial solution of $A\,y' = B_1\,y$.

Conversely, if equation (2) is of the above form, with $B_0 = -S^n$ and S is a polynomial solution of $A\,y' = B_1\,y$, it is easy to see that $\omega^i S$ $(i = 0, 1, \ldots, n-1)$ are all solutions. For, substituting $\omega^i S$ in the above equation, we have $A\,S' = B_1\,S$, which is true since S is a solution of $A\,y' = B_1\,y$. This holds independently of the values of i. *Thus* $\omega^i S$ are all solutions. This completes the proof.

Let a, b_0, b_1 be the degrees of A, B_0, B_1 respectively. The question of the existence of polynomial solutions other than $\omega^i S$ is settled by the following theorem.

THEOREM 2. If (i) $n > 2$, (ii) $B_0 = -S^n$ where S is a polynomial solution of $A\,y' = B_1\,y$, (iii) $b_1 < \frac{n-2}{n}\,b_0$ and (iv) the degree of S is not the singular exponent for equation (3), then equation (3) has no polynomial solutions other than $\omega^i S$ $(i = 0, 1, \ldots, n-1)$.

(For definition of singular exponent see [1]). We omit the proof of this result, which requires a rather involved use of the classification of the degrees of polynomial solutions as developed in [1].

We finally show that for a certain form of equation (2) there is at most one polynomial solution.

THEOREM 3. If $a - 1 < \frac{n-2}{n}\,b_0$ and $b_0 > 0$, then the equation

$$A(x)\,y' = B_0(x) + y^n \tag{4}$$

has at most one polynomial solution.

PROOF. By Theorem (2) of [3] the only possible polynomial solutions are $\omega^i S$ $(i = 0, \ldots, n-1)$. We now show that at most one of these polynomials can be a solution. By assuming two polynomials $\omega^j S$ and $\omega^k S$ $(\omega^j \neq \omega^k)$ to be solutions we get

$$A\,\omega^j S' = -Q, \qquad A\,\omega^k S' = -Q$$

whence $S' = 0$. Hence $S = C$ (constant) and $Q = 0$. This implies $B_0 = -C^n =$ constant, which is a contradiction.

REFERENCES

1. M. BHARGAVA and H. KAUFMAN. *Degrees of Polynomial Solutions of a Class of Riccati-type Differential Equations*, Collectanea Mathematica Volume XVI, 1964, pp. 211-223.

2. J. G. CAMPBELL and M. GOLOMB. *On the Polynomial Solutions of a Riccati Equation*, American Math. Monthly, Volume 61, 1954, pp. 402-404.

3. M. BHARGAVA and H. KAUFMAN. *Existence of Polynomial Solutions of a Class of Riccati-type Differential Equations*, Collectanea Mathematica, Volume XVII, 1965, pp. 135-143.

McGILL UNIVERSITY
Montreal, Canadá