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On Picard bundles over Prym varieties

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Abstract

Let $\pi: Y \to X$ be a covering between non-singular irreducible projective curves. The Jacobian $J(Y)$ has two natural subvarieties, namely, the Prym variety P and the variety $\pi^*(J(X))$. We prove that the restriction of the Picard bundle to the subvariety $\pi^*(J(X))$ is stable. Moreover, if $\widetilde{P}$ is a principally polarized Prym-Tyurin variety associated with P, we prove that the induced Abel-Prym morphism $\widetilde{\rho}: Y \to \widetilde{P}$ is birational to its image for genus $g_X > 2$ and $\deg \pi \neq 2$. We use this result to prove that the Picard bundle over the Prym variety is simple and moreover is stable when $\widetilde{\rho}$ is not birational onto its image.

Keywords: Prym varieties, stable vector bundles, Picard bundle.

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Introduction

Let Y be a non-singular irreducible projective curve of genus $g_Y \geq 2$ over an algebraically closed field of characteristic zero. Let $\mathcal{P}_Y$ be the (normalized) Poincaré bundle over $Y \times J^d(Y)$. The higher direct images of the Poincaré bundle on $J^d(Y)$ are called Picard sheaves. If $d > 2g_Y - 2$, the direct image $\mathcal{W}_J$ of $\mathcal{P}_Y$ is locally free and is called the Picard bundle. These sheaves have been studied extensively in the last years (see e.g. [10], [13] and [7]). In [7], Ein and Lazarsfeld prove the stability (with respect to the theta divisor) of the Picard bundle $\mathcal{W}_J$ when $d > 2g_Y - 1$, generalizing a result of Kempf ([10]) holding for $d = 2g_Y - 1$.

In this paper we study the restrictions of the Picard bundle $\mathcal{W}_J$ to the Prym variety associated with a covering $\pi: Y \to X$ and to the subvariety $\pi^*(J(X))$ of the Jacobian $J(Y)$. More precisely, let $\pi: Y \to X$ be a covering between non-singular irreducible projective curves. Let $(J(Y), \Theta_Y)$ be the principally polarized Jacobian of degree zero of Y and $\mathcal{P}_Y$ the Poincaré bundle over $Y \times J(Y)$. Fixing a line bundle L_0 on Y of degree $d > 2g_Y - 2$, we identify $J(Y)$ with $J^d(Y)$ and consider the Picard bundle $\mathcal{W}_J = p_{2*}(p_1^* L_0 \otimes \mathcal{P}_Y)$ on $J(Y)$. We denote the restriction of $\mathcal{W}_J$ to the subvarieties P and $\pi^*(J(X))$ of $J(Y)$ by $\mathcal{W}_P$ and $\mathcal{W}_\pi$, respectively.

We prove that if $\pi_*(L_0)$ is stable and $\deg(\pi_*(L_0)) > 2ng_X$, then $(\pi^*)^*(\mathcal{W}_J)$ is Θ_X-stable on $J(X)$, where $\pi^*: J(X) \to J(Y)$ is the pull-back morphism (see Theorem 2.2 and Remark 2.1), and deduce that $\mathcal{W}_\pi$ is stable with respect to the polarization $\Theta_\pi = \Theta_Y|_{\pi^*(J(X))}$.

The restriction Θ_P of Θ_Y to P need not be a multiple of a principal polarization. However, it is possible to construct (see [14]) a principally polarized abelian variety $(\widetilde{P}, \Xi)$ and an isogeny $f: \widetilde{P} \to P$ with $f^{-1}(\Theta_P) \equiv n\Xi$ such that there exists a map $\widetilde{\rho}: Y \to \widetilde{P}$ with $\widetilde{\rho}(Y)$ and Ξ^{m-1} numerically equivalent (where $m = \dim P = \dim \widetilde{P}$).

We prove that if $n \neq 2$ and $g_X > 2$, the morphism $\widetilde{\rho}: Y \to \widetilde{P}$ is birational to its image (Theorem 1.2) and use a property of the Fourier-Mukai transform to show that in this case $\mathcal{W}_P$ is simple. Moreover, we characterize explicitly the cases where $\widetilde{\rho}$ may not be birational. In this case, we prove that the Prym variety is the image in $J(Y)$ of the Jacobian of the normalization of the curve $\widetilde{\rho}(Y)$. As a consequence of the stability of $\mathcal{W}_\pi$, the Picard bundle $\mathcal{W}_P$ is stable with respect to $\Theta_Y|_P$ when $\widetilde{\rho}$ is not birational. In particular, for $g_X \geq 2$ and $d > 2g_Y + 4$, $\mathcal{W}_P$ is simple (Theorem 2.9).

As an application of the above results (see Theorems 3.1 and 3.3), we show that the corresponding Picard bundle $\mathcal{W}_\xi$ over the moduli space of stable vector bundles of rank n and fixed determinant ξ, with n and $d = \deg\xi$ coprime, is simple when $g_X \geq 2$ and $d > n(n+1)(g_X - 1) + 6$. In the case $g_X = 2$ and $n = 2$ the bundle $\mathcal{W}_\xi$ is stable for $d > 10$.

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1. Prym varieties

We shall denote by $J(X)$ the Jacobian of degree zero of a non-singular projective irreducible curve X and by Θ_X the natural polarization on $J(X)$ given by the Riemann theta divisor.

Let $\pi: Y \to X$ be a covering of degree n between non-singular irreducible projective curves of genus g_Y and g_X, respectively. We have the norm map $N_\pi: J(Y) \to J(X)$ and the pull back map $\pi^*: J(X) \to J(Y)$. It is known that the map $h: J(X) \to \pi^*(J(X))$ induced by π^* is an isogeny such that $h^{-1}\left(\Theta_Y|_{\pi^*(J(X))}\right) \equiv n\Theta_X$. We will denote by Θ_π the restriction of the theta divisor of $J(Y)$ to $\pi^*(J(X))$.

The Prym variety P associated with the covering is the abelian subvariety of $J(Y)$ defined by $P = \mathrm{Im}(n\mathrm{Id}_{J(Y)} - \pi^* \circ N_\pi)$. We shall denote by μ the morphism $(n\mathrm{Id}_{J(Y)} - \pi^* \circ N_\pi): J(Y) \longrightarrow P$.

On P there exists a natural polarization Θ_P given by the restriction of the theta divisor Θ_Y to P. In general Θ_P is not a multiple of a principal polarization (see [11], Theorem 3.3, pp. 376). The following theorem gathers several standard results about Prym-Tyurin varieties for which we refer to [11], [14] and [8].

Theorem 1.1

Let $\pi: Y \to X$ be a covering of degree n. There exist a principally polarized abelian variety $(\widetilde{P}, \Xi)$, an isogeny $f: \widetilde{P} \to P$ and a morphism $\widetilde{\mu}: J(Y) \to \widetilde{P}$ such that:

1. $f^{-1}(\Theta_P) \equiv n\Xi$.
2. *The map $f \circ \widetilde{\mu}: J(Y) \to P$ coincides with the morphism μ defined as above.*
3. *The restriction $\widetilde{\mu}|_P$ is surjective and the map $f \circ \widetilde{\mu}|_P$ coincides with the multiplication by n on P.*

Moreover, the following relation holds

$$N_\pi^{-1}(\Theta_X) + \widetilde{\mu}^{-1}(\Xi) \equiv n\Theta_Y.$$

The isogeny f and the morphism $\widetilde{\mu}$ in the previous theorem are not unique. From now on we shall consider a fixed isogeny $f: \widetilde{P} \to P$ enjoying the properties listed in Theorem 1.1. Let y_0 be a fixed point on Y. Let $\widetilde{\rho}_{y_0}: Y \to \widetilde{P}$ and $\rho_{y_0}: Y \to P$ be the morphisms

$$\widetilde{\rho}_{y_0} = \widetilde{\mu} \circ \alpha_{y_0} \qquad \text{and} \qquad \rho_{y_0} = f \circ \widetilde{\rho}_{y_0} = \mu \circ \alpha_{y_0}$$

where $\alpha_{y_0}: Y \to J(Y)$ is the Abel-Jacobi map $y \mapsto \mathcal{O}(y - y_0)$. We shall refer to ρ_{y_0} and $\widetilde{\rho}_{y_0}$ as *the Abel-Prym maps* of P and $\widetilde{P}$, respectively. When there is no risk of confusion we shall omit the point y_0 in the notation.

By the theory of Prym-Tyurin varieties (see [14] and [11]) the morphism $\widetilde{\rho}$ has the following property:

$$\widetilde{\rho}_*[Y] = \frac{n}{(m-1)!} \bigwedge^{m-1} [\Xi],$$

where $m = \dim P = \dim \widetilde{P}$. This last property is usually expressed by saying that the curve Y (or rather the morphism $\widetilde{\rho}$) is of class n in $(\widetilde{P}, \Xi)$.

Theorem 1.2

Let $\pi: Y \to X$ be a covering of degree n. If $g_X > 2$ and $n \neq 2$ then the Abel-Prym morphism $\widetilde{\rho}: Y \to \widetilde{P}$ is birational to its image.

Proof. Suppose that $\widetilde{\rho}$ is not birational to its image and $n \neq 2$. Let Z be the normalization of the curve $\widetilde{\rho}(Y)$ and let $\widetilde{\pi}: Y \to Z$ and $Z \to \widetilde{P}$ be the induced maps. If $\tilde{n}$ is the degree of $\widetilde{\pi}$, then $\tilde{n}$ divides n and the curve Z is of class $n/\tilde{n}$ in $\widetilde{P}$. That is,

$$\begin{array}{ccccc} & & Y & & \\ & {\scriptstyle n:1} \swarrow {\scriptstyle \pi} & & {\scriptstyle \tilde{\pi}} \searrow {\scriptstyle \tilde{n}:1} & \\ X & & & & Z \end{array}$$

Since the morphism $\tilde{\mu}$ is surjective, Z generates $\tilde{P}$. Therefore the map $\tilde{u}: J(Z) \to \widetilde{P}$ defined by the universal property of the Jacobian is surjective. Hence $g_Y - g_X = \dim \widetilde{P} \leq g_Z$, where g_Z is the genus of Z.

By the Riemann-Hurwitz formula for the coverings π and $\tilde{\pi}$, we have the following inequality

$$g_Y \leq \frac{g_Y - 1 - (\mathrm{deg}R_{\pi})/2}{n} + \frac{g_Y - 1 - (\mathrm{deg}R_{\tilde{\pi}})/2}{\tilde{n}} + 2 \tag{1}$$

where R_{π} and $R_{\tilde{\pi}}$ are the ramification divisors.

Since $\tilde{n}$ divides n, denote by $q \in \mathbb{N}$ the quotient $n/\tilde{n}$. Hence, the inequality (1) implies that

$$(n - q - 1)g_Y \leq 2n - q - 1 - \left(\frac{(\mathrm{deg}R_{\pi})}{2} + \frac{q(\mathrm{deg}R_{\tilde{\pi}})}{2} \right).$$

Since $((\mathrm{deg}R_{\pi})/2 + q(\mathrm{deg}R_{\tilde{\pi}})/2) \geq 0$ and $n \neq q + 1$ because $n \neq 2$, we have

$$g_Y \leq 2 + \left[\frac{q+1}{n-q-1} \right]$$

where $[-]$ denotes the integral part of a rational number. Since $\tilde{n} > 1$, it is easy to check that $\left[\frac{q+1}{n-q-1} \right] \geq 1$ if and only if $\tilde{n} = 2, 3, 4$.

Studying these special cases we obtain that if $\widetilde{\rho}$ is not birational and $n \neq 2$, then only the following cases can occur:

$$n = 3, \quad \tilde{n} = 3 \text{ and } g_Y \leq 4; \qquad n = 4, \begin{cases} \text{if } \tilde{n} = 4 & \text{then } g_Y \leq 3 \\ \text{if } \tilde{n} = 2 & \text{then } g_Y \leq 5 \end{cases};$$

$$n = 6, \begin{cases} \text{if } \tilde{n} = 6 & \text{then } g_Y \leq 2 \\ \text{if } \tilde{n} = 3 & \text{then } g_Y \leq 3 \\ \text{if } \tilde{n} = 2 & \text{then } g_Y \leq 4 \end{cases}; \qquad n \neq 3, 4, 6, \begin{cases} \text{if } \tilde{n} = 2 & \text{then } g_Y \leq 3 \\ \text{if } \tilde{n} \neq 2 & \text{then } g_Y \leq 2 \end{cases}.$$

From this it follows that, if $\widetilde{\rho}$ in not birational to its image and $n \neq 2$, then $g_Y \leq 5$.

Now suppose that $g_X \geq 2$ (and $n \neq 2$). By the Riemann-Hurwitz formula applied to the covering π, it follows that if the Abel-Prym $\widetilde{\rho}$ is not birational to its image then necessarily only the two cases given by the following numerical conditions can occur:

$$\begin{array}{llllll} n = 3, & \tilde{n} = 3, & g_Y = 4, & g_X = 2, & g_Z = 2, & R_{\pi} = R_{\tilde{\pi}} = 0 \\ n = 4, & \tilde{n} = 2, & g_Y = 5, & g_X = 2, & g_Z = 3, & R_{\pi} = R_{\tilde{\pi}} = 0 \end{array} \tag{2}$$

Therefore, if we assume that $g_X > 2$ and $n \neq 2$, the theorem is proved. □

Remark 1.3. From the proof of Theorem 1.2 we deduce that if $g_X \geq 2$, then the Abel-Prym map $\widetilde{\rho}$ may possibly not be birational to its image in the cases given in (2) and when $n = 2$. For $n = 2$, it is well-known that if Y is not an hyperelliptic curve then ρ is birational to its image and therefore $\widetilde{\rho}$ is birational to its image.

We conclude this section with a lemma that we shall use later on.

Lemma 1.4

For any $\tilde{\xi} \in \widetilde{P}$, we have

$$\widetilde{\rho}^* \left(\tau_{\tilde{\xi}}^* \mathcal{O}_{\widetilde{P}}(\Xi) \right) \cong \widetilde{\rho}^* \mathcal{O}_{\widetilde{P}}(\Xi) \otimes f(\widetilde{\xi})^{-1},$$

where $\tau_{\tilde{\xi}}$ is the translation by $\tilde{\xi}$.

Proof. By Theorem 1.1, there exists $\xi \in P$ such that $\tilde{\mu}(\xi) = \tilde{\xi}$. Using the relation between the polarizations given in that theorem, we obtain

$$\begin{aligned}\widetilde{\rho}^* \left(\tau_{\tilde{\xi}}^* \mathcal{O}_{\widetilde{P}}(\Xi) \right) &\cong \alpha^* \left(\widetilde{\mu}^* \left(\tau_{\tilde{\xi}}^* \mathcal{O}_{\widetilde{P}}(\Xi) \right) \right) \cong \alpha^* \left(\tau_{\xi}^* \left(\widetilde{\mu}^* \mathcal{O}_{\widetilde{P}}(\Xi) \right) \right) \\ &\cong \alpha^* \left(\tau_{\xi}^* \left(\mathcal{O}_{J(Y)}(n\Theta_Y) \otimes N_\pi^* \mathcal{O}_{J(X)}(\Theta_X)^\vee \otimes \mathcal{N} \right) \right),\end{aligned}$$

where $\mathcal{N} \in \operatorname{Pic}^0(J(Y))$. In particular, $\mathcal{N}$ is invariant under translation. Moreover, we have $\tau_\xi^* \left(N_\pi^* \mathcal{O}_{J(X)}(\Theta_X)^\vee \right) \cong N_\pi^* \left(\mathcal{O}_{J(X)}(\Theta_X)^\vee \right)$ because $\xi \in P$ and

$$\alpha^* \left(\tau_\xi^* \mathcal{O}_{J(Y)}(n\Theta_Y) \right) \cong \alpha^* \left(\mathcal{O}_{J(Y)}(n\Theta_Y) \right) \otimes \xi^{-n}.$$

Therefore, it follows that $\widetilde{\rho}^* \left(\tau_{\tilde{\xi}}^* \mathcal{O}_{\widetilde{P}}(\Xi) \right) \cong \widetilde{\rho}^* \mathcal{O}_{\widetilde{P}}(\Xi) \otimes \xi^{-n}$. Since $\xi^{-n} = f(\widetilde{\mu}|_P(\xi^{-1})) = f(\tilde{\xi})^{-1}$, the lemma is proved. □

2. Stability and restrictions of Picard bundles

Let Y be a non-singular irreducible projective curve of genus $g_Y \geq 2$. We fix a line bundle L_0 on Y of degree d and a point $y_0 \in Y$. Let $\mathcal{P}_Y$ be the Poincaré bundle over $Y \times J(Y)$ normalized with respect to the point y_0, i.e. $\mathcal{P}_Y|_{\{y_0\} \times J(Y)} \cong \mathcal{O}_{J(Y)}$. The Picard sheaves (relative to L_0) on $J(Y)$ are defined as the higher direct images $R^i p_{2*}(p_1^* L_0 \otimes \mathcal{P}_Y)$ where p_j are the canonical projections of $Y \times J(Y)$ to the jth factor. If $d > 2g_Y - 2$, then $\mathcal{W}_J := p_{2*}(p_1^* L_0 \otimes \mathcal{P}_Y)$ is a vector bundle, known as *the Picard bundle*. We shall consider the restrictions of $\mathcal{W}_J$ to some subvarieties of $J(Y)$.

Let $\pi\colon Y \to X$ be a covering of degree n between non-singular projective irreducible curves of genus g_Y and g_X respectively, and let $\pi^*\colon J(X) \to J(Y)$ be the pull-back morphism. As in §1 we consider the Prym variety (P, Θ_P) and $(\pi^*(J(X)), \Theta_\pi)$ in $J(Y)$.

From now on we fix a line bundle L_0 on Y of degree $d > 2g_Y - 2$. Denote by $\mathcal{W}_P$ and $\mathcal{W}_\pi$ the restrictions of the Picard bundle $\mathcal{W}_J$ (relative to L_0) to P and $\pi^*(J(X))$, respectively.

Remark 2.1. We can take a line bundle $L_0 \in J^d(Y)$ such that $\pi_*(L_0)$ is stable and $\deg \pi_*(L_0) > 2ng_X$. Indeed, for a generic line bundle L on Y, A. Beauville proved in [3] that π_*L is stable on X if $|\chi(L)| \leq g_X + \frac{g_X^2}{n}$ or $\deg\pi < \max\{g_X(1+\sqrt{3})-1, 2g_X+2\}$. Therefore, if $L \in J^{d'}(Y)$ with

$$g_Y - g_X - 1 - \frac{g_X^2}{n} \leq d' \leq g_Y + g_X - 1 + \frac{g_X^2}{n}, \tag{3}$$

then π_*L and $\pi_*(L \otimes \pi^* M)$ are stable for any line bundle M on X. Thus, for a generic line bundle $L_0 \in J^d(Y)$ with d such that:

(i) $d > \max\{2g_Y - 2 + 2n - (\deg R_\pi)/2\ ,\ 2g_Y - 2\}$,

(ii) d is equivalent (modulo n) to a number d' such that d' fulfils the condition (3),

we have that $\pi_*(L_0)$ is stable and $\deg \pi_*(L_0) = \deg L_0 - (\deg R_\pi)/2 > 2ng_X$. In the case $\deg\pi < \max\{g_X(1+\sqrt{3})-1, 2g_X+2\}$, it is sufficient that $\deg L_0$ fulfils the condition (i).

Theorem 2.2

If $\pi_*(L_0) = E_0$ is stable and $\deg(E_0) > 2ng_X$, then $(\pi^*)^*(\mathcal{W}_J)$ is Θ_X-stable on $J(X)$.

Proof. For convenience of writing, we set up the following commutative diagram:

$$\begin{array}{ccccc}
X & \xleftarrow{\ \pi\ } & Y & & \\
\uparrow{\scriptstyle p_X} & & {\scriptstyle q_1}\uparrow & & \nwarrow{\scriptstyle p_1} \\
X \times J(X) & \xleftarrow{\pi \times id} & Y \times J(X) & \xrightarrow{id \times \pi^*} & Y \times J(Y) \\
{\scriptstyle p_{J(X)}}\searrow & & \downarrow{\scriptstyle q_2} & & {\scriptstyle p_2}\downarrow \\
 & & J(X) & \xrightarrow{\ \pi^*\ } & J(Y)
\end{array} \tag{4}$$

where the vertical morphisms are the natural projections.

Fix a Poincaré bundle $\mathcal{P}_X$ on $X \times J(X)$ normalized with respect to the point $\pi(y_0) \in X$. Then, the vector bundle $(id \times \pi^*)^*\mathcal{P}_Y$ on $Y \times J(X)$ is isomorphic to the bundle $(\pi \times id)^*\mathcal{P}_X$. From diagram (4) and the base change formula, we have

$$\begin{aligned}(\pi^*)^*(\mathcal{W}_J) &\cong q_{2*}\left(q_1^*L_0 \otimes (id \times \pi^*)^*\mathcal{P}_Y\right) \cong p_{J(X)*}\left((\pi \times id)_*\left(q_1^*L_0 \otimes (\pi \times id)^*\mathcal{P}_X\right)\right) \\ &\cong p_{J(X)*}\left((\pi \times id)_*(q_1^*L_0) \otimes \mathcal{P}_X\right) \cong p_{J(X)*}(p_X^*(E_0) \otimes \mathcal{P}_X).\end{aligned}$$

Since E_0 is stable of degree $d > 2ng_X$, $p_X^*(E_0) \otimes \mathcal{P}_X$ is a family of stable bundles parametrized by $J(X)$. Such family corresponds to an embedding of the Jacobian in the moduli space $\mathcal{M}(n,d)$ of stable vector bundles of rank n and degree d. As in the proof of Theorem 2.5 in [12], it follows that $p_{J(X)*}(p_X^*(E_0) \otimes \mathcal{P}_X)$ is Θ_X-stable. □

Corollary 2.3

If $\pi_*(L_0) = E_0$ is stable and $\deg(E_0) > 2ng_X$, then the restriction $\mathcal{W}_\pi$ of $\mathcal{W}_J$ to $\pi^*(J(X))$ is Θ_π-stable.

Proof. The map $h : J(X) \to \pi^*(J(X))$ is an isogeny such that $h^{-1}(\Theta_\pi) \equiv n\Theta_X$. Since $h^*(\mathcal{W}_\pi) \cong (\pi^*)^*(\mathcal{W}_J)$, it follows from Lemma 2.1 in [1] that $\mathcal{W}_\pi$ is Θ_π-stable. $\square$

To study the restriction $\mathcal{W}_P$ of $\mathcal{W}_J$ to the Prym variety P we consider two cases, namely when the Abel-Prym map $\widetilde{\rho}$ is not birational and when it is. In both cases we can reduce our study to consider the corresponding vector bundle over $(\widetilde{P}, \Xi)$ where $f : \widetilde{P} \to P$ is a fixed isogeny as in Theorem 1.1. That is, if $j_P : P \hookrightarrow J(Y)$ is the natural inclusion, we shall denote by $\beta : \widetilde{P} \to J(Y)$ the composition map $f \circ j_P$ and by $\mathcal{W}_{\widetilde{P}}$ the vector bundle $\beta^*(\mathcal{W}_J)$. Actually, $f^*(\mathcal{W}_P) \cong \mathcal{W}_{\widetilde{P}}$.

Proposition 2.4

If $\mathcal{W}_{\widetilde{P}}$ is Ξ-stable (resp. simple), then $\mathcal{W}_P$ is Θ_P-stable (resp. simple).

Proof. The stability follows from Lemma 2.1 in [1]. Suppose $\mathcal{W}_{\widetilde{P}}$ is simple. Since $\mathcal{O}_P \hookrightarrow f_*\mathcal{O}_{\widetilde{P}}$ is injective, the map

$$\begin{aligned}\mathrm{Hom}(\mathcal{W}_P, \mathcal{W}_P) \hookrightarrow \mathrm{Hom}(\mathcal{W}_P, \mathcal{W}_P \otimes f_*\mathcal{O}_{\widetilde{P}}) &\cong \mathrm{Hom}(\mathcal{W}_P, f_*f^*\mathcal{W}_P)\\ &\cong \mathrm{Hom}(\mathcal{W}_{\widetilde{P}}, \mathcal{W}_{\widetilde{P}}) \cong \mathbb{C}\end{aligned}$$

is injective. Hence, $\mathrm{Hom}(\mathcal{W}_P, \mathcal{W}_P) \cong \mathbb{C}$ and $\mathcal{W}_P$ is simple. $\square$

Suppose $\widetilde{\rho}$ is not birational. As in section § 1, let Z be the normalization of $\widetilde{\rho}(Y)$ and $\tilde{\pi} : Y \to Z$ the morphism induced by $\widetilde{\rho}$ (of degree $\tilde{n}$).

Proposition 2.5

If the Abel-Prym map $\widetilde{\rho}\colon Y \to \widetilde{P}$ is not birational to its image and $\deg L_0 > 2g_Y + 4$, *then the Picard bundle $\mathcal{W}_P$ is Θ_P-stable.*

Proof. We consider separately each case where $\widetilde{\rho}$ might not be birational (see Remark 1.3).

1) Suppose $n = 2$.

We have that $\deg \tilde{\pi} = 2$ and $\widetilde{\rho}(Y)$ is of class one in $(\widetilde{P}, \Xi)$. Then, by Matsusaka's Criterion, $Z = \widetilde{\rho}(Y)$ and $(J(Z), \Theta_Z) \cong (\widetilde{P}, \Xi)$.

From the construction of the map $\widetilde{\rho}$, we have $N_{\tilde{\pi}} = \tilde{\mu}$. Moreover, $\tilde{\mu}^t = j_P \circ f = \beta$ (see section § 1 in [14]). Therefore the map $\tilde{\pi}^* : J(Z) \to J(Y)$ coincides with the map $\beta : \widetilde{P} \to J(Y)$ via the isomorphism $J(Z) \cong \widetilde{P}$. By definition of $\mathcal{W}_{\widetilde{P}}$, we have

$$\mathcal{W}_{\widetilde{P}} \cong (\widetilde{\pi}^*)^*(\mathcal{W}_J).$$

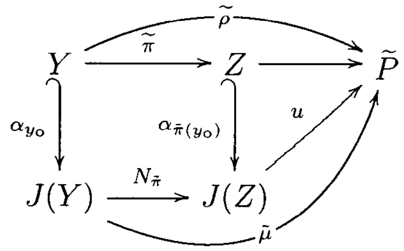
In this case, $\deg \tilde{\pi} = 2 < \max\{g_Z(1 + \sqrt{3}) - 1, 2g_Z + 2\}$ because $g_Z \geq 1$. Therefore, by Remark 2.1 and Theorem 2.2, if $\deg L_0 = d > 2g_Y + 2 \geq \max\{2g_Y + 2 - \deg(R_{\tilde{\pi}})/2, 2g_Y - 2\}$, then $(\widetilde{\pi}^*)^*(\mathcal{W}_J) \cong \mathcal{W}_{\widetilde{P}}$ is Ξ-stable and from Proposition 2.4, $\mathcal{W}_P$ is Θ_P-stable.

2) Suppose $n = 3$, $\tilde{n} = 3$, $g_Y = 4$, $g_X = 2$, $g_Z = 2$, $R_\pi = R_{\tilde{\pi}} = 0$.

Since $\widetilde{\rho}(Y)$ is of class one in $(\widetilde{P}, \Xi)$, $Z = \widetilde{\rho}(Y)$ and $(J(Z), \Theta_Z) \cong (\widetilde{P}, \Xi)$. In this case, $\deg \tilde{\pi} = 3 < \max\{g_Z(1+\sqrt{3}) - 1, 2g_Z + 2\}$. Hence, if $\deg L_0 = d > 2g_Y + 4 = 12$, then $\mathcal{W}_{\widetilde{P}}$ is Ξ-stable and $\mathcal{W}_P$ is Θ_P-stable.

3) Suppose $n = 4$, $\tilde{n} = 2$, $g_Y = 5$, $g_X = 2$, $g_Z = 3$, $R_\pi = R_{\tilde{\pi}} = 0$.

We have the following commutative diagram:



where u is the map induced by $Z \to \widetilde{P}$. Therefore,

$$\beta = j_P \circ f = \tilde{\mu}^t = \tilde{\pi}^* \circ u^t \colon \widetilde{P} \longrightarrow J(Y)$$

and $\mathrm{Im}\tilde{\mu}^t = P \subset \mathrm{Im}\tilde{\pi}^*$. Since $\dim P = 3 = \dim J(Z) = \dim \tilde{\pi}^*(J(Z))$, it follows that $\tilde{\pi}^*(J(Z)) = P \subset J(Y)$. Since $\deg\tilde{\pi} = 2 < \max\{g_Z(1+\sqrt{3}) - 1, 2g_Z + 2\}$, if $\deg L_0 = d > 2g_Y + 2 = 12$ then, from Corollary 2.3, the restriction of $\mathcal{W}_J$ to $\tilde{\pi}^*(J(Z))$ is $\Theta_Y|_{\tilde{\pi}^*(J(Z))}$-stable, i.e. W_P is Θ_P-stable. □

Remark 2.6. Observe that in the previous proposition the result holds for any line bundle L_0 (of degree $d > 2g_Y + 4$) whereas in Remark 2.1 the condition "$\pi_*(L_0)$ is stable" holds for a generic line bundle L_0. If the Picard bundle $\mathcal{W}_{L_0,P}$ corresponding to L_0 is stable, then $\mathcal{W}_{L,P} = p_{2*}(p_1^* L \otimes \mathcal{P}_Y|_{Y\times P})$ is stable for any L (of degree d). In fact, since $L \otimes L_0^{-1} \in J(Y)$, one can write $L \cong L_0 \otimes \pi^* N \otimes M$ where $M \in P$ and $N \in J(X)$, therefore the two Picard bundles are related by $\mathcal{W}_{L,P} \cong \tau_M^*(\mathcal{W}_{L_0 \otimes \pi^* N, P})$, where $\tau_M \colon P \to P$ is the translation by M. Since $\pi_*(L_0)$ is stable, $\pi_*(L_0 \otimes \pi^* N)$ is stable as well. Hence, $\mathcal{W}_{L,P}$ is stable.

Before considering the case when the map $\widetilde{\rho}$ is birational we shall describe the bundle $\mathcal{W}_{\widetilde{P}}$ in a different way.

The abelian variety $(\widetilde{P}, \Xi)$ is principally polarized, so that we can identify $\widetilde{P}$ with its dual abelian variety. The Poincaré bundle on $\widetilde{P} \times \widetilde{P}$ is given by

$$\mathcal{Q} \cong m^* \mathcal{O}_{\widetilde{P}}(\Xi) \otimes q_1^* \mathcal{O}_{\widetilde{P}}(\Xi)^\vee \otimes q_2^* \mathcal{O}_{\widetilde{P}}(\Xi)^\vee,$$

where m is the multiplication law on $\widetilde{P}$ and q_i the canonical projections of $\widetilde{P} \times \widetilde{P}$ to the i-th factor.

Proposition 2.7

The bundle $\mathcal{W}_{\widetilde{P}}$ is isomorphic to the bundle $q_{2*}(q_1^*(\widetilde{\rho}_*(L_0)) \otimes \mathcal{Q}^\vee)$.

Proof. Consider the commutative diagram:

$$\begin{array}{ccccc} & & Y & \xrightarrow{\widetilde{\rho}} & \widetilde{P} \\ & & {\scriptstyle p_1}\uparrow & & \uparrow{\scriptstyle q_1} \\ Y \times P & \xleftarrow{id \times f} & Y \times \tilde{P} & \xrightarrow{\tilde{\rho} \times id} & \tilde{P} \times \tilde{P} \\ & & {\scriptstyle p_2} \searrow & & \downarrow{\scriptstyle q_2} \\ & & & & \tilde{P} \end{array}$$

Let $\mathcal{P}_{\widetilde{P}}$ be the line bundle $\mathcal{P}_{\widetilde{P}} = (id \times (f \circ j_P))^*(\mathcal{P}_Y)$ on $Y \times \widetilde{P}$, where $j_P : P \to J(X)$ is the natural inclusion. From the normalization of $\mathcal{P}_Y$, the restrictions of the line bundles $\mathcal{P}_{\widetilde{P}}$ and $(\widetilde{\rho} \times id)^* \mathcal{Q}^\vee$ to $\{y_0\} \times \widetilde{P}$ are trivial and for every $\tilde{\xi} \in \widetilde{P}$, the restrictions of both bundles to $Y \times \{\tilde{\xi}\}$ are isomorphic to $f(\tilde{\xi})$ by Lemma 1.4. Therefore, $\mathcal{P}_{\widetilde{P}} \cong (\widetilde{\rho} \times id)^* \mathcal{Q}^\vee$.

From the projection formula and the base-change formula, we have

$$\begin{aligned} \mathcal{W}_{\widetilde{P}} &\cong p_{2*}\left(p_1^* L_0 \otimes \mathcal{P}_{\widetilde{P}}\right) \cong q_{2*}\left((\widetilde{\rho} \times id)_* \left(p_1^* L_0 \otimes (\widetilde{\rho} \times id)^* \mathcal{Q}^\vee\right)\right) \\ &\cong q_{2*}\left((\widetilde{\rho} \times id)_*(p_1^* L_0) \otimes \mathcal{Q}^\vee\right) \cong q_{2*}\left(q_1^*\left(\widetilde{\rho}_*(L_0)\right) \otimes \mathcal{Q}^\vee\right). \ \square \end{aligned}$$

Proposition 2.8

If the Abel-Prym map $\widetilde{\rho}$ is birational to its image then the restriction $\mathcal{W}_P$ of the Picard bundle $\mathcal{W}_J$ to the Prym variety is simple.

Proof. From Proposition 2.4 it is enough to prove that $\mathcal{W}_{\widetilde{P}}$ is simple. The bundle $\mathcal{W}_{\widetilde{P}}$ is the Fourier-Mukai transform of the sheaf $\widetilde{\rho}_*(L_0)$ by Proposition 2.7. Since $\deg(L_0) > 2g_Y - 2$, the sheaf $q_1^*(\widetilde{\rho}_*(L_0)) \otimes \mathcal{Q}^\vee$ has only one non-null direct image. Then the Fourier-Mukai transform is a complex concentrated in degree zero and, therefore, $\mathcal{W}_{\widetilde{P}}$ is simple if $\widetilde{\rho}_*(L_0)$ is simple (see [13], Corollary 2.5).

To prove that $\widetilde{\rho}_*(L_0)$ is simple we write $\widetilde{\rho}$ as a composite $j \circ \tilde{\nu}$ where $\tilde{\nu}: Y \to \widetilde{\rho}(Y)$ is a birational morphism between curves and $j: \widetilde{\rho}(Y) \to \widetilde{P}$ is a closed embedding. Since $\widetilde{\rho}$ is birational to its image, $\tilde{\nu}_*(L_0)$ is of rank 1. Therefore the sheaf K defined by

$$0 \to K \to \tilde{\nu}^* \tilde{\nu}_* L_0 \to L_0 \to 0$$

is a torsion sheaf. From the exact sequence

$$0 \to \mathrm{Hom}(L_0, L_0) \to \mathrm{Hom}(\tilde{\nu}^* \tilde{\nu}_* L_0, L_0) \to \mathrm{Hom}(K, L_0)$$

and the fact that $\mathrm{Hom}(K, L_0) = 0$, we obtain

$$\mathrm{Hom}(\tilde{\nu}_* L_0, \tilde{\nu}_* L_0) \cong \mathrm{Hom}(\tilde{\nu}^* \tilde{\nu}_* L_0, L_0) \cong \mathrm{Hom}(L_0, L_0) \cong \mathbb{C}.$$

Since j is a closed embedding, $j^* j_* E \cong E$ for any sheaf E, and applying the adjunction formula again, we obtain that $j_* \tilde{\nu}_*(L_0) \cong \widetilde{\rho}_* L_0$ is simple. $\square$

From Proposition 2.5 and Proposition 2.8 we have

Theorem 2.9

The restriction $\mathcal{W}_P$ of the Picard bundle $\mathcal{W}_J$ to the Prym variety is simple if $g_X \geq 2$ and $\deg L_0 > 2g_Y + 4$.

3. Applications

We shall recall the construction and properties of spectral coverings given by Beauville, Narasimhan and Ramanan in [4].

Let X be a non-singular projective irreducible curve of genus $g_X \geq 2$. Denote by $\mathcal{M}(n,d)$ the moduli space of stable vector bundles over X of rank n and degree d. If n and d are coprime there exists a universal family $\mathcal{U}$ parametrized by $\mathcal{M}(n,d)$ called the Poincaré bundle. If $d > 2n(g_X - 1)$ the Picard bundle $\mathcal{W}$ is the direct image of $\mathcal{U}$ and is locally free. We denote by $\mathcal{W}_\xi$ the restriction of $\mathcal{W}$ to the subvariety $\mathcal{M}_\xi \subset \mathcal{M}(n,d)$ determined by stable bundles with fixed determinant $\xi \in J^d(X)$.

Let K be the canonical bundle over X and $W = \bigoplus_{i=1}^n H^0(X, K^i)$. For every element $s = (s_1, \ldots, s_n) \in W$, we denote by Y_s the associated spectral curve (see [4]). For a general $s \in W$, Y_s is non-singular of genus $g_{Y_s} = n^2(g_X - 1) + 1$ and the morphism

$$\pi_s : Y_s \longrightarrow X$$

is of degree n.

In [4] it is proved that if $\delta = d + n(n-1)(g_X - 1)$ then there is an open subvariety T_δ of $J^\delta(Y_s)$ such that the morphism $T_\delta \to \mathcal{M}(n,d)$ defined by $L \mapsto \pi_{s*}(L)$ is dominant. Moreover, the direct image induces a dominant rational map

$$f : P' \dashrightarrow \mathcal{M}_\xi$$

defined on an open subvariety $T' \subseteq P'$, where P' is a translate of the Prym variety P_s of π_s (see [4], Proposition 5.7). The complement of the open subvariety $T' \subseteq P'$ is of codimension at least 2. Actually, $f : T' \longrightarrow \mathcal{M}_\xi$ is generically finite.

Consider the following commutative diagram:

$$\begin{array}{ccccc} Y_s \times T' & \xrightarrow{\pi_s \times id} & X \times T' & \xrightarrow{id \times f} & X \times \mathcal{M}_\xi \\ & {}_{q_2'}\searrow & \downarrow{\scriptstyle q_2} & & \downarrow{\scriptstyle q_2} \\ & & T' & \xrightarrow{f} & \mathcal{M}_\xi \end{array}$$

where q_2', q_2, p_2 are the projections to the second factor.

If $\mathcal{P}_{T'}$ is the restriction of the Poincaré bundle over $Y_s \times P'$ to $Y_s \times T'$ and $\mathcal{U}_\xi$ is the restriction of the universal bundle $\mathcal{U}$ to $X \times \mathcal{M}_\xi$, then, by the definition of f,

$$(id \times f)^*(\mathcal{U}_\xi) \cong (\pi_s \times id)_*(\mathcal{P}_{T'}) \otimes q_2^*(M)$$

for some line bundle M over T', which depends on the choice of $\mathcal{U}$. Therefore,

$$f^*(\mathcal{W}_\xi) \cong (q_2)_*(id \times f)^*(\mathcal{U}_\xi) \cong (q_2)_*((\pi_s \times id)_*(\mathcal{P}_{T'})) \otimes M \cong (q_2')_*(\mathcal{P}_{T'}) \otimes M.$$

Actually, $(q_2')_*(\mathcal{P}_{T'})$ is just the restriction of the Picard bundle $\mathcal{W}_{P'}$ over P' to T'.

Theorem 3.1

The Picard bundle $\mathcal{W}_\xi$ on $\mathcal{M}_\xi$ is simple for $d > n(n+1)(g_X - 1) + 6$ and $g_X \geq 2$.

Proof. Since $\operatorname{codim}(T') \geq 2$, we have

$$\operatorname{End}(\mathcal{W}_{P'}) \cong H^0(P', \mathcal{E}nd(\mathcal{W}_{P'})) \cong H^0(T', \mathcal{E}nd(\mathcal{W}_{P'}|_{T'})) \cong \operatorname{End}(f^*\mathcal{W}_\xi).$$

The Abel-Prym map for the spectral cover is birational to its image if $n \neq 2$ or $g_X > 2$. In this case, from Proposition 2.8, the bundle $\mathcal{W}_{P'}$ is simple if $\delta = d + n(n-1)(g_X - 1) > 2g_{Y_s} - 2$. Since the map $f: T' \to \mathcal{M}_\xi$ is dominant and generically finite, as in the proof of Proposition 2.4, we deduce that the Picard bundle $\mathcal{W}_\xi$ on $\mathcal{M}_\xi$ is simple for degree $d > n(n+1)(g_X - 1)$.

For $n = 2$ and $g_X = 2$, if the map $\widetilde{\rho}$ is not birational to its image, then by the proof of Proposition 2.5, $\mathcal{W}_\xi$ is simple for $d > n(n+1)(g_X - 1) + 6$. □

Remark 3.2. Denote by Θ_ξ a generalized theta divisor on $\mathcal{M}_\xi$. By [12], Theorem 4.3, we have that

$$f^*(\mathcal{O}(\Theta_\xi)) \cong \mathcal{O}(\Theta_{P'})|_{T'}$$

where $\Theta_{P'}$ is the restriction of the theta divisor of $J^\delta(Y_s)$ to P'. Since $\mathcal{W}_{P'}|_{T'} \cong f^*\mathcal{W}_\xi$, by Lemma 2.1 in [1], it follows that if $\mathcal{W}_{P'}$ is $\Theta_{P'}$-stable, then $\mathcal{W}_\xi$ is Θ_ξ-stable. Moreover, from Theorem 2.2 in [6], $\mathcal{W}_{P'}$ is stable if $\widetilde{\rho}_L^*(\mathcal{W}_{P'})$ is stable on Y for a general line bundle L.

Now we will focus on the case $n = 2$ and $g_X = 2$. $\mathcal{M}(2, \xi)$ will denote the moduli space of stable rank 2 vector bundles with fixed determinant ξ (with degξ odd). We shall show how to construct a spectral covering $\pi: Y_s \to X$ of degree 2 in such way that the curve Y_s is hyperelliptic. Let $p: X \to \mathbb{P}^1$ be the covering of degree 2 given by the canonical bundle K_X. Let s be a section of $H^0(X, K_X^2)$ such that the spectral covering $\pi: Y_s \to X$ of degree 2 corresponding to $(0, s)$ is smooth, integral and such that the induced map from the Prym variety associated to $Y_s \to X$ to the moduli space $\mathcal{M}(2, \xi)$ is dominant. Observe that since $g_X = 2$ we can write the section s as a product $s = s_1 \cdot s_2$ where $s_i \in H^0(X, K_X)$ for $i = 1, 2$. If $P_i + Q_i$ is the divisor associated to the section s_i, then $p(P_i) = p(Q_i) = z_i \in \mathbb{P}^1$.

According to the construction of cyclic coverings of curves given in [9], the morphism $\bar{X} = \mathbb{P}^1 \to \mathbb{P}^1$ of degree 2 ramified at the points z_1, z_2 is the cyclic covering defined by the construction data $(D = z_1 + z_2, L = \mathcal{O}_{\mathbb{P}^1}(2))$. Let C be the desingularization of the curve $X \times_{\mathbb{P}^1} \bar{X}$. From Theorem 2.13 in [9] the map $C \to X$ is the cyclic covering associated to the data $(p^{-1}(D), p^*L)$, but since $p^{-1}(D) = \{P_1, Q_1, P_2, Q_2\}$ and $p^*L \cong K_X^2$, then $Y_s \cong C$ and the map $Y_s \cong C \to \bar{X} = \mathbb{P}^1$ is of degree 2. Therefore the curve Y_s is hyperelliptic.

Let P be the Prym variety associated to the degree 2 covering $Y_s \to X$. Since the spectral curve Y_s is hyperelliptic then the Abel-Prym morphism $\rho: Y_s \to P$ is not birational to its image. As in the case 3) of the proof of Proposition 2.5, it follows that in this case if $\delta = d + n(n-1)(g_X - 1) > 2g_{Y_s} + 2 = 12$, then the Picard bundle on the Prym variety P is stable with respect to the restriction of the theta divisor. From Remark 3.2 the stability of Picard bundles on $\mathcal{M}_X(2, \xi)$ is ensured. Thus we have proved the the following result.

Theorem 3.3

Let X be a smooth projective irreducible curve of genus 2 and let $\mathcal{M}_X(2, \xi)$ be the moduli space of of rank 2 stable vector bundles on X with fixed determinant ξ. If $d = \deg(\xi) > 10$ and d odd, then the Picard bundle $\mathcal{W}_\xi$ on $\mathcal{M}_X(2, \xi)$ is stable with respect to the theta divisor.

Remark 3.4. After this paper was completed, I. Biswas and T. Gómez informed us that they have obtained the stability of the Picard bundle in the case of the moduli space $\mathcal{M}_X(2,\xi)$ of rank 2 stable bundles over a smooth curve X of genus $g_X \geq 3$ such that $d = \deg\xi \geq 4g_X - 3$ and odd ([5]).

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