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Boundary controllability of a chain of serially connected Euler-Bernoulli beams with interior masses

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Abstract

The aim is to study the boundary controllability of a system modelling the vibrations of a network of N Euler-Bernoulli beams serially connected by $(N-1)$ vibrating interior point masses. Using the classical Hilbert Uniqueness Method, the control problem is reduced to the obtention of an observability inequality. The solution is then expressed in terms of Fourier series so that one of the sufficient conditions for the observability inequality is that the distance between two consecutive large eigenvalues of the spatial operator involved in this evolution problem is superior to a minimal fixed value. This property called spectral gap holds. It is proved using the exterior matrix method due to W.H. Paulsen. Two more asymptotic estimates involving the eigenfunctions are required for the observability inequality to hold. They are established using an adequate basis.

1. Introduction

In the last few years various physical models of multi-link flexible structures consisting of finitely many interconnected flexible elements such as strings, beams, plates, shells have been mathematically studied. See [11, 12, 17, 26, 28] for instance. The spectral analysis of such structures has some applications to control or stabilization problems ([26, 27]). For interconnected strings (corresponding to a second-order operator on each string), a lot of results have been obtained: the asymptotic behaviour of

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the eigenvalues ([1, 2, 10, 37]), the relationship between the eigenvalues and algebraic theory (cf. [7, 8, 26, 36]), qualitative properties of solutions (see [10, 40]) and finally studies of the Green function (cf. [22, 41, 43]).

For interconnected beams (corresponding to a fourth-order operator on each beam), some results on the asymptotic behaviour of the eigenvalues and on the relationship between the eigenvalues and algebraic theory were obtained by Nicaise and Dekoninck in [19, 20, 21] with different kinds of connections using the method developed by von Below in [7] to get the characteristic equation associated to the eigenvalues.

The authors used the same method in a recent paper ([34]) to compute the spectrum for a hybrid system of N flexible beams connected by n vibrating point masses. This type of structure was studied by Castro and Zuazua in many papers (see [13, 14, 15, 16, 18]) and Castro and Hansen ([24]). They have restricted themselves to the case of two beams applying their results on the spectral theory to controllability. They have shown that if the constant of rotational inertia is positive, due to the presence of the mass, the system is well-posed in asymmetric spaces (spaces with different regularity on both sides of the mass) and consequently, the space of controllable data is also asymmetric. For a vanishing constant of rotational inertia the system is not well-posed in asymmetric spaces and the presence of the point mass does not affect the controllability of the system.

Note that S.W. Taylor proved similar results at the same time in [44] using different techniques based on the method presented in [30] for exact controllability.

In a second paper the authors investigated the same problem as in [34] but with different methods which are more adapted to the study of controllability. The way they computed the spectrum in [34] was too complicated to get results about boundary controllability which is also our point here. Using the classical Hilbert Uniqueness Method, the control problem was reduced to the obtention of an observability inequality. The solution was then expressed in terms of Fourier series so that it is also enough to show that the distance between two consecutive large eigenvalues of the spatial operator involved in the evolution problem is superior to a minimal fixed value. This property called spectral gap holds as soon as the roots of a function denoted by f_∞ (and giving the asymptotic behaviour of the eigenvalues) are all simple. For a network of $N = 2$ different beams, this assumption on the multiplicity of the roots of f_∞ (denoted by (A)) was proved to be satisfied and controllability followed. For higher values of N, a numerical approach allowed one to prove (A) in many situations and no counterexample had been found but the problem of giving a general proof of controllability had remained open.

The aim of this paper is to give a definite answer to this problem using the technique of exterior matrices due to W.H. Paulsen (presented for other purposes in [39]) and already used in the same type of context by D. Mercier in [33]. The author studies the same problem as will be done in the following but without interior point masses.

The authors have also been working on transmission problems on networks for a few years: Mercier studied in [32] transmission problems for elliptic systems in the sense of Agmon-Douglis-Nirenberg on polygonal networks with general boundary and interface conditions.

In [5], Régnier and Ali Mehmeti studied the spectral solution of a one-dimensional Klein-Gordon transmission problem corresponding to a particle submitted to a potential step and interpreted the phase gap between the original and reflected term in the tunnel effect case as a delay in the reflection of the particle. At the same time in [42], Régnier extended this technique to a two-dimensional problem which had been first studied from a spectral point of view by Croc and Dermenjian.

Let us finally quote the paper by Nicaise and Valein ([38]) on stabilization of the one-dimensional wave equation with a delay term in the feedbacks. They use the same method as we did in a previous paper [34] (technique developed by von Below in [7]) to get the characteristic equation associated to the eigenvalues and apply this spectral analysis to stabilization.

In this paper we will still investigate the same problem as in [34, 35] but adding the exterior matrices method to solve the technical problems remaining in the previous papers. Moreover, a particular network is considered which is a chain of N serially connected branches ($N \geq 2$) with $n = N + 1$ vertices (denoted by E_i) such that the $(N-1)$ interior vertices are point masses with mass M_i.

Let us recall the control problem (PC):

$$\begin{cases} u_{j,tt}(x,t) + a_j u_{jx_j^{(4)}}(x,t) = 0, \ \forall j \in \{1,\dots,N\} \\ u_{j,tt}(E_i,t) - \dfrac{1}{M_i}\left(\displaystyle\sum_{j\in N_i} a_j \frac{\partial^3 u_j}{\partial \nu_j^3}(E_i)\right) = 0, \ \forall j \in \{1,\dots,N\}, \ \forall i \in I_{\text{int}} \\ u_j(E_i) = z_i, \ \forall i \in I_{\text{int}}, \forall j \in N_i \\ \displaystyle\sum_{j\in N_i} \frac{\partial u_j}{\partial \nu_j}(E_i) = 0, \forall i \in I_{\text{int}} \\ a_l \dfrac{\partial^2 u_l}{\partial \nu_l^2}(E_i) = a_j \dfrac{\partial^2 u_j}{\partial \nu_j^2}(E_i), \ \forall i \in I_{\text{int}}, \forall (l,j) \in N_i^2 \\ u_j(E_i) = 0, \ \forall i \in I_{\text{ext}}, \ \forall j \in N_i \\ \dfrac{\partial^2 u_j}{\partial \nu_j^2}(E_i) = 0, \ \forall i \in I_{\text{ext}} - \{i_0\}, \ \forall j \in N_i \text{ and } \dfrac{\partial^2 u_j}{\partial \nu_j^2}(E_{i_0}) = q, \ \forall j \in N_{i_0}. \end{cases}$$

The scalar functions $u_j(x,t)$ and $z_i(t)$ contain the information on the vertical displacements of the beams ($1 \leq j \leq N$) and of the point masses ($1 \leq i \leq N-1$). These displacements are described by the first two equations where the a_j's are mechanical constants, I_{int} (respectively I_{ext}) is the set of indices corresponding to the interior (resp. exterior) vertices of the network, N_i is the set of edges adjacent to the vertex E_i.

The third, fourth and fifth equations are transmission conditions. The sixth and seventh ones are boundary conditions.

Note that the control function $q = q(t)$ acts on the system through the exterior node E_{i_0} on the quantity $\dfrac{\partial^2 u_j}{\partial \nu_j^2}$.

The problem of exact controllability can be formulated as follows: *for any time $T > 0$, find the class $\mathcal{H}$ of initial conditions for which there exists a control function q*

in $L^2(0,T)$ *such that the solution of Problem* (PC) *is at rest at time* $t = T$ *i.e.*

$$\begin{cases} u_j(x,T) = 0,\ \forall x \in k_j,\ j \in \{1,\ldots,N\} \\ u_{j,t}(x,T) = 0,\ \forall x \in k_j,\ j \in \{1,\ldots,N\} \end{cases}$$

and

$$\begin{cases} z_i(T) = 0,\ \forall i \in I_{\text{int}} \\ z_{i,t}(T) = 0,\ \forall i \in I_{\text{int}} \end{cases}$$

(k_j denotes the j-th edge of the network.)

Before starting to study the core of the problem, we apply in Section 2 the terminology of networks to our particular network. The whole terminology can be found in early contributions of Lumer and Gramsch (cf. [31, 23]) as well as in papers by Ali Mehmeti ([3, 4]), von Below ([7]) and Nicaise ([36, 6]) in the eighties. We also recall some properties of the spatial operator $\mathcal{A}$ involved in the considered evolution problem (cf. Lemma 1).

In Section 3, we recall that it is enough to get an observability inequality for controllability to hold. This classical result is an application of the Hilbert Uniqueness Method to our situation (see Lemma 3). A result due to Haraux (cf. [25]) is recalled which states that it is sufficient for the spectrum of the operator to have a particular asymptotic behaviour (called spectral gap) to get the required observability inequality provided that two additional estimates also hold.

The study of the asymptotic behaviour of the eigenvalues of the spatial operator $\mathcal{A}$ is thus envisaged in Section 4. This behaviour is given by that of the roots of a function called f_∞. In order to avoid the cancellation of the large order terms which occurred in [34], the characteristic equation is computed using the exterior matrix method due to Paulsen ([39]) and already used by D. Mercier in [33]. The so-called spectral gap property is thus satisfied for any chain of N serially connected beams with interior point masses.

Two estimates involving the first derivative of an eigenfunction of the problem at the node where the control acts as well as a norm of the same eigenfunction remain to be proved. It is the aim of Section 5. The choice of the basis h_i (cf. Section 5.1) in which the eigenfunctions are decomposed is crucial for the asymptotic behaviour of the eigenfunctions to be studied since the expressions are very complicated especially for large values of N. In particular the exponential factor in h_3 has an important role since its presence keeps the exponential terms from being disseminated in the different matrices which would not allow an easy estimation of the involved quantities as λ tends to infinity.

The last section contains the main theorem which states the controllability of the problem. The general case which was not accessible via the techniques used in [34, 35] is solved.

2. Preliminaries

2.1 Notation

A particular network is considered here, which is a chain of N branches ($N \geq 2$) as

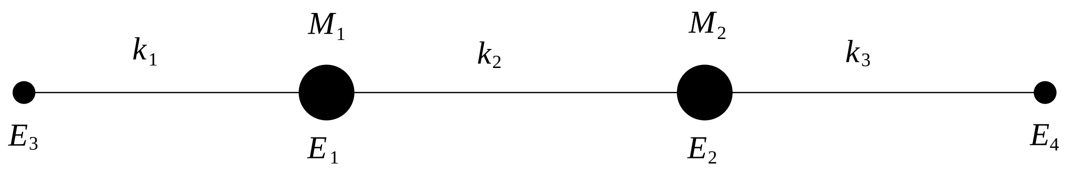

Figure 1: Graph with N=3 edges

represented above for $N = 3$. This means with the usual terminology of networks (introduced in [9] and recalled in [20, 34] for example) that the graph has N edges (denoted by k_j) and $N+1$ vertices (denoted by E_i).

The interior vertices are: $E_1, \cdots, E_{N-1}$ and the exterior vertices are E_N and E_{N+1}.

For shortness, we later on denote by I_{int} (respectively I_{ext}) the set of indices corresponding to the interior (resp. exterior) vertices i.e. $I_{\text{int}} = \{1, \cdots, N-1\}$ and $I_{\text{ext}} = \{N; N+1\}$.

For each vertex E_i, we also denote by $N_i = \{j \in \{1, ..., N\} \, : E_i \in k_j\}$ the set of edges adjacent to E_i.

The network is $G = \bigcup_{j=1}^{N} k_j$.

2.2 Data and framework

Following Castro and Zuazua ([16]), we study a linear system modelling the vibrations of beams connected by point masses but with N beams (instead of two) and $N-1$ point masses. For each edge k_j (representing a beam of our network of beams), we fix mechanical constants: $m_j > 0$ (the mass density of the beam k_j) and $E_j I_j > 0$ (the flexural rigidity of k_j). We set $a_j = \dfrac{E_j I_j}{m_j}$. For each interior vertex E_i, $i \in I_{\text{int}}$, we fix the mass $M_i > 0$ $(1 \le i \le N-1)$.

So the scalar functions $u_j(x,t)$ and $z_i(t)$ for $x \in G$ and $t > 0$ contain the information on the vertical displacements of the beams $(1 \le j \le N)$ and of the point masses $(1 \le i \le N-1)$. Our aim is to study the spectrum of the spatial operator (involved in the evolution problem) which is defined as follows.

First define the inner product $(\cdot,\cdot)_H$ on $H := \prod_{j=1}^{N} L^2((0,l_j)) \times \mathbb{R}^{N-1}$ by

$$((u,z),(w,s))_H = \sum_{j=1}^{N} \int_0^{l_j} u_j(x_j) w_j(x_j) dx_j + \sum_{i=1}^{N-1} M_i z_i s_i \,. \tag{1}$$

And define the operator $\mathcal{A}$ on the Hilbert space H endowed with the above inner product, by:

$$\begin{cases} D(\mathcal{A}) = \{(u,z) \in H \; : \; u_j \in H^4((0,l_j)) \text{ satisfying (3) to (7) hereafter}\} \\ \forall (u,z) \in D(\mathcal{A}), \; \mathcal{A}(u,z) = \left(\left(a_j u_{j x_j^{(4)}} \right)_{j=1}^{N}, -\dfrac{1}{M_i} \Big(\displaystyle\sum_{j \in N_i} a_j \frac{\partial^3 u_j}{\partial \nu_j^3}(E_i) \Big)_{i=1}^{N-1} \right) \end{cases} \tag{2}$$

where $\dfrac{\partial u_j}{\partial \nu_j}(E_i)$ means the exterior normal derivative of u_j at E_i.

$$u_j(E_i) = z_i,\ \forall i \in I_{\text{int}}, \forall j \in N_i \tag{3}$$

$$\sum_{j\in N_i} \frac{\partial u_j}{\partial \nu_j}(E_i) = 0, \forall i \in I_{\text{int}} \tag{4}$$

$$a_l \frac{\partial^2 u_l}{\partial \nu_l^2}(E_i) = a_j \frac{\partial^2 u_j}{\partial \nu_j^2}(E_i),\ \forall i \in I_{\text{int}}, \forall (l,j) \in N_i^2 \tag{5}$$

$$u_j(E_i) = 0,\ \forall i \in I_{\text{ext}},\ \forall j \in N_i \tag{6}$$

$$\frac{\partial^2 u_j}{\partial \nu_j^2}(E_i) = 0,\ \forall i \in I_{\text{ext}},\ \forall j \in N_i. \tag{7}$$

Notice that the conditions (3) imply the continuity of u on G. The conditions (4) and (5) are transmission conditions at the interior nodes and (6) and (7) are boundary conditions.

Lemma 1 (Properties of the operator $\mathcal{A}$).

The operator $\mathcal{A}$ defined by (2) is a nonnegative self-adjoint operator with a compact resolvent.

Proof. The reason for $\mathcal{A}$ to be a self-adjoint operator with a compact resolvent, is that it is the Friedrichs extension of the triple (H, V, a) defined by

$$V = \left\{ U = (u, z) \in \prod_{j=1}^{N} H^2((0, l_j)) \times \mathbb{R}^{N-1} \ : \text{satisfying} \quad (3),\ (4),\ (6) \right\}$$

which is a Hilbert space endowed with the inner product

$$(U, W)_V = ((u, z), (w, s))_V = \sum_{j=1}^{N} (u_j, w_j)_{H^2((0,l_j))} + \sum_{i=1}^{N-1} M_i z_i s_i$$

where $(.,.)_{H^2((0,l_j))}$ is the usual inner product on $(0, l_j)$ and

$$a(U, W) = \sum_{j=1}^{N} a_j \int_0^{l_j} u_{jx_j^{(2)}}(x_j) w_{jx_j^{(2)}}(x_j) dx_j \tag{8}$$

cf. [34] for the details of the proof. □

3. General results about controllability applied to a chain of N Euler-Bernoulli beams

3.1 Controllability and observability

Let us first recall the definition of controllability applied to the problem we will consider. Then we classically establish a sufficient condition called observability inequality.

Let i_0 be an element of I_{ext} and (PC) be the following problem:

$$\begin{cases} u_{j,tt}(x,t) + a_j u_{jx_j^{(4)}}(x,t) = 0,\ \forall j \in \{1,\dots,N\} \\ u_{j,tt}(E_i,t) - \dfrac{1}{M_i}\Big(\displaystyle\sum_{j\in N_i} a_j \frac{\partial^3 u_j}{\partial \nu_j^3}(E_i)\Big) = 0,\ \forall j \in \{1,\dots,N\},\ \forall i \in I_{\text{int}} \\ u_j(E_i) = z_i,\ \forall i \in I_{int}, \forall j \in N_i \\ \displaystyle\sum_{j\in N_i} \frac{\partial u_j}{\partial \nu_j}(E_i) = 0, \forall i \in I_{\text{int}} \\ a_l \dfrac{\partial^2 u_l}{\partial \nu_l^2}(E_i) = a_j \dfrac{\partial^2 u_j}{\partial \nu_j^2}(E_i),\ \forall i \in I_{\text{int}}, \forall (l,j) \in N_i^2 \\ u_j(E_i) = 0,\ \forall i \in I_{\text{ext}},\ \forall j \in N_i \\ \dfrac{\partial^2 u_j}{\partial \nu_j^2}(E_i) = 0,\ \forall i \in I_{\text{ext}} - \{i_0\},\ \forall j \in N_i \text{ and } \dfrac{\partial^2 u_j}{\partial \nu_j^2}(E_{i_0}) = q,\ \forall j \in N_{i_0}\,. \end{cases}$$

Note that the control function $q = q(t)$ acts on the system through the exterior node E_{i_0} on the quantity $\dfrac{\partial^2 u_j}{\partial \nu_j^2}$.

Definition 2 (Controllability). Problem (PC) is exactly controllable at time $T > 0$ if there exists q in $L^2(0,T)$ such that the solution (u,z) of the above problem (PC) satisfies:

$$\begin{cases} u_j(x,T) = 0,\ \forall x \in k_j,\ j \in \{1,\dots,N\} \\ u_{j,t}(x,T) = 0,\ \forall x \in k_j,\ j \in \{1,\dots,N\} \end{cases}$$

and

$$\begin{cases} z_i(T) = 0,\ \forall i \in I_{\text{int}} \\ z_{i,t}(T) = 0,\ \forall i \in I_{\text{int}}\,. \end{cases}$$

The aim is then to find a class $\mathcal{H}$ of initial conditions $U^0 = (u^0, z^0)$, $U^1 = (u^1, z^1)$ such that Problem (PC) is controllable (recall that U^0 and U^1 are still used for $(u(0), z(0))$ and $(u_t(0), z_t(0))$ respectively as in [35, Lemma 5]).

Using the classical Hilbert Uniqueness Method (HUM) developed in Lions (cf. [29]) leads to the following sufficient condition:

Lemma 3 (Observability inequality and controllability).

Let $T > 0$. A sufficient condition for Problem (PC) to be controllable at time T is the existence of two strictly positive constants κ_1 and κ_2 such that, if $(U^0, U^1) \in \mathcal{H}$,

$$\kappa_1 \cdot \|(U^0,U^1)\|^2_{\mathcal{H}} \le \int_0^T \left| a_j \frac{\partial u_j}{\partial \nu_j}(E_{i_0},t)\right|^2 dt \le \kappa_2 \cdot \|(U^0,U^1)\|^2_{\mathcal{H}} \tag{9}$$

with $j \in N_{i_0}$.

Note that N_{i_0} only contains one element since E_{i_0} is an exterior node and see a previous paper by the authors ([35]) for the proof.

In the following N_{i_0} is chosen to contain only k_1 and the control acts on the system through the exterior node E_N. In fact we could have chosen the other exterior node E_{N+1} since the exterior nodes play a symmetric role. It would not change the way we solve the problem.

This first analysis of the problem is a generalization of what Castro and Zuazua do in [16]. The observability inequality (9) was proved there with the space $\mathcal{H} = \mathcal{H}_{1/4}$ (defined in [16]) in the case of a network with two beams connected by a point mass. It is (5.2) in [16, Proposition 3]. To prove that inequality, the authors used the properties of the eigenvalues. We will generalize this approach to the case of a chain of N branches in the following sections.

3.2 Observability inequality and spectral gap

Since the solution is expressed in terms of Fourier series (cf. [35, Proposition 6]), the observability inequality will be proved using the following result due to Haraux (cf. [25]) and also used by Castro and Zuazua (cf. [16]).

Lemma 4 (Observability inequality and spectral gap).

Let λ_n be a sequence of real numbers such that there exist $(\alpha, \beta, N_0) \in \mathbb{R}^2 \times \mathbb{N}$ satisfying

$$\lambda_{n+1} - \lambda_n \geq \alpha > 0, \forall |n| \geq N_0 \tag{10}$$

and $\lambda_{n+1} - \lambda_n \geq \beta > 0$.

Consider also $T > \pi/\alpha$. Then there exist two constants $C_1(T)$ and $C_2(T)$ which only depend on α, β and N_0 such that, if $f(t) = \sum_{n \in Z} \alpha_n e^{i\lambda_n t}$

$$C_1(T) \sum_{n\in Z} |\alpha_n|^2 \leq \int_{-T}^{T} |f(t)|^2 dt \leq C_2(T) \sum_{n \in Z} |\alpha_n|^2$$

for all $(\alpha_n) \in l^2(\mathbb{R})$.

Thus the aim of the following two sections is to prove the spectral gap (10) on the first hand and the following estimate for any eigenfunction ϕ associated to the eigenvalue λ^2 on the second hand (the eigenvalue problem associated to Problem (P) is recalled in the following section): there exist positive constants K_1 and K_2 such that for large values of λ

$$K_1 \cdot \|\phi\|_H^2 \cdot \lambda \leq |\phi_1'(0)|^2 \leq K_2 \cdot \|\phi\|_H^2 \cdot \lambda \tag{11}$$

with the norm $\|\cdot\|_H$ defined by (1).

4. Proof of the spectral gap using exterior matrices

In order to establish the spectral gap (10), we need to determine the asymptotic behaviour of the characteristic equation of the eigenvalue problem associated to Problem (P). This problem can be written as: $\lambda^2 \in \sigma(\mathcal{A})$ $(\lambda > 0)$ is an eigenvalue of $\mathcal{A}$ with associated eigenvector $\Phi = (\phi, z) \in D(\mathcal{A})$ if and only if ϕ satisfies the transmission and boundary conditions (3)-(7) of Section 2 and

$$(EP) \left\{ \begin{array}{ll} a_j \phi_{jx_j^{(4)}} = \lambda^2 \phi_j & \text{on } (0, \mathrm{l}_j), \ \forall \ \mathrm{j} \in \ \{1, ..., \mathrm{N}\} \\ \displaystyle\sum_{j \in N_i} a_j \frac{\partial^3 \phi_j}{\partial \nu_j^3}(\mathrm{E}_i) = \lambda^2 M_i z_i, & \forall i \in I_{\text{int}} \\ \phi_j \in H^4((0, l_j)), & \forall j \in \{1, ..., N\}. \end{array} \right.$$

The characteristic equation was computed in a previous paper by the authors (cf. [35]). The notation will be recalled since they have been slightly modified in order to simplify the expressions.

4.1 Recall of notation and of some properties

Let ϕ be a non-trivial solution of the above eigenvalue problem (EP) and λ^2 $(\lambda > 0)$ be the corresponding eigenvalue.

For each $j \in \{1, ..., N\}$, the vector function V_j is defined by

$$V_j(x) = \left(\phi_j(x), \frac{a_j \phi_{jx_j^{(2)}}(x)}{\lambda}, \frac{\phi_{jx_j^{(1)}}(x)}{\sqrt{\lambda}}, \frac{a_j \phi_{jx_j^{(3)}}(x)}{\lambda^{3/2}} \right)^t, \forall x \in [0, l_j].$$

Note that the divisions by λ, $\sqrt{\lambda}$ and $\lambda^{3/2}$ will simplify the expressions later on. That is why they have been introduced (compared to the previous papers [35, 33]). The increasing order of derivation is not used here so that we get the restriction of a matrix in the characteristic equation (13).

Keeping the notation a_j and l_j introduced in Section 2, the matrix A_j is $A_j = A(a_j, b_j)$ with $b_j = a_j^{-1/4} l_j$ and $A(a, b)$ the square matrix of order 4 defined by

$$\frac{1}{4}\begin{pmatrix} e^{b\sqrt{\lambda}} + e^{-b\sqrt{\lambda}} + 2c & \frac{e^{b\sqrt{\lambda}} + e^{-b\sqrt{\lambda}} - 2c}{a^{3/2}} & \frac{e^{b\sqrt{\lambda}} - e^{-b\sqrt{\lambda}} + 2s}{a^{1/4}} & \frac{e^{b\sqrt{\lambda}} - e^{-b\sqrt{\lambda}} - 2s}{a^{7/4}} \\ a^{3/2}(e^{b\sqrt{\lambda}} + e^{-b\sqrt{\lambda}} - 2c) & e^{b\sqrt{\lambda}} + e^{-b\sqrt{\lambda}} + 2c & a^{5/4}(e^{b\sqrt{\lambda}} - e^{-b\sqrt{\lambda}} - 2s) & \frac{e^{b\sqrt{\lambda}} - e^{-b\sqrt{\lambda}} + 2s}{a^{1/4}} \\ a^{1/4}(e^{b\sqrt{\lambda}} - e^{-b\sqrt{\lambda}} - 2s) & \frac{e^{b\sqrt{\lambda}} - e^{-b\sqrt{\lambda}} + 2s}{a^{5/4}} & e^{b\sqrt{\lambda}} + e^{-b\sqrt{\lambda}} + 2c & \frac{e^{b\sqrt{\lambda}} + e^{-b\sqrt{\lambda}} - 2c}{a^{3/2}} \\ a^{7/4}(e^{b\sqrt{\lambda}} - e^{-b\sqrt{\lambda}} + 2s) & a^{1/4}(e^{b\sqrt{\lambda}} - e^{-b\sqrt{\lambda}} - 2s) & a^{3/2}(e^{b\sqrt{\lambda}} + e^{-b\sqrt{\lambda}} - 2c) & e^{b\sqrt{\lambda}} + e^{-b\sqrt{\lambda}} + 2c \end{pmatrix}$$

with the notation $c = \cos(b\sqrt{\lambda})$, $s = \sin(b\sqrt{\lambda})$.

The matrix T_j depends on the interior masses M_j (cf. Section 2) and on the eigenvalue λ^2 in the following way:

$$T_j = T(M_j, \lambda) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sqrt{\lambda} M_j & 0 & 0 & 1 \end{pmatrix}.$$

Note that the vector function V_j has been changed (compared to the previous papers [35, 33]) which changes the term $\lambda^2 M_j$ into $\sqrt{\lambda} M_j$ in the above matrix.

To finish with, the matrix $M(\lambda)$ is given by

$$M(\lambda) = A_N T_{N-1} A_{N-1} ... A_2 T_1 A_1. \tag{12}$$

Lemma 5 (A few trivial but useful properties)

With the notation introduced above:

$$\begin{aligned} &V_j(l_j) = A_j V_j(0),\ \forall j \in \{1, ..., N\} \\ &V_{j+1}(0) = T_j V_j(l_j),\ \forall j \in \{1, ..., N-1\} \\ &V_N(l_N) = M(\lambda) V_1(0)\,. \end{aligned}$$

The proof is given in [35].

Theorem 6 (The characteristic equation for the eigenvalue problem corresponding to a chain of N branches).

$\lambda^2 > 0$ is an eigenvalue of $\mathcal{A}$ if and only if λ satisfies the characteristic equation

$$f(\sqrt{\lambda}) = \det(M_{12}(\lambda)) = 0, \tag{13}$$

where $M_{12}(\lambda)$ is the square matrix of order 2 which is the restriction of the matrix $M(\lambda)$, given by (12), to its first two lines and its last two columns.

For that property again, the proof is given in [35].

4.2 Rewriting of the characteristic equation using the exterior matrix method

The exterior matrix method presented in [39] is a very useful method which allows to compute asymptotically the eigenfrequencies for the vibrations of serially connected elements which are governed by fourth-order equations. But our goal is to get the spectral gap. The main idea is to exploit the special properties of the exterior matrices associated to our problem in order to obtain the desired results.

The whole section makes use of the same ideas as in a previous paper by D. Mercier ([33]).

First, we simply recall the definition of exterior matrix and some useful results that we need in the sequel (see [39] for more details).

DEFINITION 7 If $M = (m_{ij})$ is a 4×4 matrix, then the exterior matrix of M is the 6×6 matrix given by: ext(M) =

$$\begin{pmatrix}
\begin{vmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{vmatrix} & \begin{vmatrix} m_{13} & m_{12} \\ m_{21} & m_{23} \end{vmatrix} & \begin{vmatrix} m_{11} & m_{14} \\ m_{21} & m_{24} \end{vmatrix} & \begin{vmatrix} m_{13} & m_{14} \\ m_{23} & m_{24} \end{vmatrix} & -\begin{vmatrix} m_{12} & m_{14} \\ m_{22} & m_{24} \end{vmatrix} & \begin{vmatrix} m_{12} & m_{13} \\ m_{22} & m_{23} \end{vmatrix} \\
\begin{vmatrix} m_{11} & m_{12} \\ m_{31} & m_{32} \end{vmatrix} & \begin{vmatrix} m_{11} & m_{13} \\ m_{31} & m_{33} \end{vmatrix} & \begin{vmatrix} m_{11} & m_{14} \\ m_{31} & m_{34} \end{vmatrix} & \begin{vmatrix} m_{13} & m_{14} \\ m_{33} & m_{34} \end{vmatrix} & -\begin{vmatrix} m_{12} & m_{14} \\ m_{32} & m_{34} \end{vmatrix} & \begin{vmatrix} m_{12} & m_{14} \\ m_{32} & m_{33} \end{vmatrix} \\
\begin{vmatrix} m_{11} & m_{12} \\ m_{41} & m_{22} \end{vmatrix} & \begin{vmatrix} m_{13} & m_{12} \\ m_{41} & m_{23} \end{vmatrix} & \begin{vmatrix} m_{11} & m_{14} \\ m_{41} & m_{24} \end{vmatrix} & \begin{vmatrix} m_{13} & m_{14} \\ m_{43} & m_{24} \end{vmatrix} & -\begin{vmatrix} m_{12} & m_{14} \\ m_{42} & m_{24} \end{vmatrix} & \begin{vmatrix} m_{12} & m_{13} \\ m_{42} & m_{23} \end{vmatrix} \\
\begin{vmatrix} m_{31} & m_{12} \\ m_{41} & m_{22} \end{vmatrix} & \begin{vmatrix} m_{33} & m_{12} \\ m_{41} & m_{23} \end{vmatrix} & \begin{vmatrix} m_{31} & m_{14} \\ m_{41} & m_{24} \end{vmatrix} & \begin{vmatrix} m_{33} & m_{14} \\ m_{43} & m_{24} \end{vmatrix} & -\begin{vmatrix} m_{32} & m_{14} \\ m_{42} & m_{24} \end{vmatrix} & \begin{vmatrix} m_{32} & m_{13} \\ m_{42} & m_{23} \end{vmatrix} \\
-\begin{vmatrix} m_{21} & m_{22} \\ m_{41} & m_{42} \end{vmatrix} & -\begin{vmatrix} m_{21} & m_{23} \\ m_{41} & m_{43} \end{vmatrix} & -\begin{vmatrix} m_{21} & m_{24} \\ m_{41} & m_{44} \end{vmatrix} & -\begin{vmatrix} m_{23} & m_{24} \\ m_{43} & m_{44} \end{vmatrix} & \begin{vmatrix} m_{22} & m_{24} \\ m_{42} & m_{44} \end{vmatrix} & -\begin{vmatrix} m_{22} & m_{23} \\ m_{42} & m_{43} \end{vmatrix} \\
\begin{vmatrix} m_{21} & m_{22} \\ m_{31} & m_{32} \end{vmatrix} & \begin{vmatrix} m_{21} & m_{23} \\ m_{31} & m_{33} \end{vmatrix} & \begin{vmatrix} m_{21} & m_{24} \\ m_{31} & m_{34} \end{vmatrix} & \begin{vmatrix} m_{23} & m_{24} \\ m_{33} & m_{34} \end{vmatrix} & -\begin{vmatrix} m_{22} & m_{24} \\ m_{32} & m_{34} \end{vmatrix} & \begin{vmatrix} m_{22} & m_{23} \\ m_{32} & m_{33} \end{vmatrix}
\end{pmatrix}$$

Lemma 8

If M_1 and M_2 are 4×4 matrices, then

$$\operatorname{ext}(M_1 M_2) = \operatorname{ext}(M_1)\operatorname{ext}(M_2).$$

Proof. See [39, Lemma 1]. □

Theorem 9 (The characteristic equation rewritten in terms of exterior matrices).
Let $\lambda^2 > 0$ be an eigenvalue of $\mathcal{A}$ then λ satisfies the characteristic equation

$$f(\sqrt{\lambda}) = e_1^t \operatorname{ext}(M(\lambda)) e_4 = 0, \tag{14}$$

or equivalently

$$f(\sqrt{\lambda}) = e_1^t \operatorname{ext}(A_N)\operatorname{ext}(T_{N-1}) ... \operatorname{ext}(A_1) e_4 = 0, \tag{15}$$

where $M(\lambda)$ is the square matrix of order 4 given by (12), $e_1^t = (1, 0, 0, 0, 0, 0)$ and $e_4^t = (0, 0, 0, 1, 0, 0)$.

Proof. Let F and B be the boundary matrices conditions given by

$$F = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \; B = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Then, (13) may be expressed equivalently as follows:

$$f(\sqrt{\lambda}) = \det(F M(\lambda) B) = 0.$$

Hence (14) is a direct consequence of [39, Proposition 1]. Moreover, (15) directly comes from (12) and the application of Lemma 8. □

Remark 10 As it was pointed out in [39], even if equation (13) gives a way of finding the eigenfrequencies, there are serious problems numerically. The final determinant typically causes the large order terms to cancel. This means that calculating (13) via a decimal approximation would be unreliable. Obviously the same problem remains when we want to analyse the asymptotic behaviour of the spectrum. In [34] we saw that the asymptotic analysis of (13) was difficult because calculation is very complicated even for small values of N (i.e. $N = 3$) and with the help of softwares such as *Mathematica.* The main advantage of (15) is that it is a way to compute the determinant before the matrices are multiplied together, so that the major cancellation occurs first.

4.3 The asymptotic behaviour of the characteristic equation

As in [33], we study the asymptotic behaviour of the exterior matrices involved in that of $M(\lambda)$ in order to get the asymptotic behaviour of the characteristic equation as λ

tends to ∞. This is enough to establish the property called spectral gap (10) since it concerns large values of λ.

Lemma 11 (Asymptotic behaviour of the characteristic equation).

Let us denote by s and H_{12}^j the following 3×3 matrices:

$$s=\begin{pmatrix}0 & -1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 0\end{pmatrix} \text{ and } H_{12}^j=\frac{1}{4}\begin{pmatrix}\frac{2s_j}{a_j^{1/2}} & \frac{c_j}{a_j^{7/4}}-\frac{s_j}{a_j^{7/4}} & -\frac{c_j}{a_j^{1/4}}-\frac{s_j}{a_j^{1/4}}\\ -\frac{c_j}{a_j^{7/4}}+\frac{s_j}{a_j^{7/4}} & \frac{c_j}{a_j^{3}} & -\frac{s_j}{a_j^{3/2}}\\ \frac{c_j}{a_j^{1/4}}+\frac{s_j}{a_j^{1/4}} & -\frac{s_j}{a_j^{3/2}} & -c_j\end{pmatrix}$$

for $j\in\{1,\cdots,N\}$ with the notation $c_j=\cos(b_j\sqrt{\lambda})$, $s_j=\sin(b_j\sqrt{\lambda})$.

Assume that the characteristic equation is still given by Theorem 9. Then there exist three constants $C\neq 0$, C' and C'' which are independent of the variable λ such that:

$$f\left(\sqrt{\lambda}\right)=C\cdot\lambda^{C'}\cdot e^{C''\sqrt{\lambda}}\cdot\left(f_\infty(\sqrt{\lambda})+g(\sqrt{\lambda})\right)$$

where

$$f_\infty\left(\sqrt{\lambda}\right)=e_1^t\left(H_{12}^N s H_{12}^{N-1} s\cdots sH_{12}^1\right)e_1 \tag{16}$$

and $e_1^t=(1,0,0)$. The function g satisfies $\lim_{\lambda\to+\infty} g(\sqrt{\lambda})=0$.

The constants are $C:=\left(\prod_{i=1}^{N-1} M_i\right)$, $C':=\dfrac{N-1}{2}$ and $C'':=\sum_{j=1}^{N} b_j$.

Thus, the asymptotic behaviour of the spectrum $\sigma(\mathcal{A})$ corresponds to the roots of the asymptotic characteristic equation

$$f_\infty\left(\sqrt{\lambda}\right)=e_1^t\left(H_{12}^N s H_{12}^{N-2} s\cdots sH_{12}^1\right)e_1=0. \tag{17}$$

Proof. The aim of the proof is to study the asymptotic behaviour of each exterior matrix contained in the expression (12). In the following the notation $o(\lambda)$ is used for a square matrix of the appropriate size such that all its terms are dominated by the function $\lambda\mapsto\lambda$ asymptotically.

After some computation, the exterior matrix of T_j defined in Section 4.1 is shown to be the 6×6 block matrix:

$$T_j=\sqrt{\lambda}M_j\begin{pmatrix}0 & 0\\ s & 0\end{pmatrix}+o(\sqrt{\lambda})$$

with s the 3×3 matrix defined in the above lemma. And the exterior matrix of A_j is

$$\operatorname{ext}(A_j)=e^{b_j\sqrt{\lambda}}H^j+o(e^{b_j\sqrt{\lambda}})$$

where H^j is the 6×6 block matrix: $H^j=\begin{pmatrix}H_{11}^j & H_{12}^j\\ H_{21}^j & H_{22}^j\end{pmatrix}$.

The block H_{12}^j is defined just above. The other blocks will not be used in the following.

Thus the exterior matrix of $M(\lambda)$ is

$$\begin{aligned}\mathrm{ext}(M(\lambda)) = \left(\sqrt{\lambda}\right)^{N-1}\left(\prod_{i=1}^{N-1} M_i\right)e^{B\sqrt{\lambda}}H^{N-1}\begin{pmatrix} 0 & 0 \\ s & 0 \end{pmatrix}H^{N-2}\cdots\begin{pmatrix} 0 & 0 \\ s & 0 \end{pmatrix}H^1 \\ + \ o\left(\left(\sqrt{\lambda}\right)^{N-1}\left(\prod_{i=1}^{N-1} M_i\right)e^{B\sqrt{\lambda}}\right)\end{aligned}$$

with $B := \sum_{j=1}^{N} b_j$. Lemma 11 follows using (15) and calculating the multiplication of the block matrices. □

Lemma 12

The roots of f_∞ are all simple. Moreover, there exists a constant $d > 0$ (which depends only on the material constants) such that for any root x_0 of f_∞

$$|f'_\infty(x_0)| \geq d. \tag{18}$$

Proof.

- The first step is to get an expression for $Q_{N-1}Q_{N-2}\cdots Q_1e_1$ by induction, where $Q_j := sH_{12}^j$ i.e.

$$Q_j = \frac{1}{4}\begin{pmatrix} \dfrac{c_j}{a_j^{7/4}} - \dfrac{s_j}{a_j^{7/4}} & -\dfrac{c_j}{a_j^3} & -\dfrac{s_j}{a_j^{3/2}} \\ \dfrac{2s_j}{a_j^{1/2}} & \dfrac{c_j}{a_j^{7/4}} - \dfrac{s_j}{a_j^{7/4}} & -\dfrac{c_j}{a_j^{1/4}} - \dfrac{s_j}{a_j^{1/4}} \\ 0 & 0 & 0 \end{pmatrix}$$

  with c_j and s_j as in Lemma 11.

  Thus $Q_1e_1 = f_1(\sqrt{\lambda})e_1 + g_1(\sqrt{\lambda})e_2$ with $f_1(x) = \dfrac{\cos(b_1x) - \sin(b_1x)}{4a_1^{7/4}}$ and $g_1(x) = \dfrac{\sin(b_1x)}{2a_1^{1/2}}$. The Wronskian of f_1 and g_1 is equal to the strictly positive constant $\dfrac{b_1}{8a_1^{9/4}}$ for any real x.

  (Recall that, by Wronskian of f_1 and g_1, we mean: $W(f_1, g_1)(x) := f_1'(x)g_1(x) - f_1(x)g_1'(x)$.)

  Assume that $Q_jQ_{j-1}\cdots Q_1e_1 = f_j(\sqrt{\lambda})e_1 + g_j(\sqrt{\lambda})e_2$ with the Wronskian of f_j and g_j strictly positive for any real x.

  Thus $Q_{j+1}\left(Q_jQ_{j-1}\cdots Q_1e_1\right) = f_j(\sqrt{\lambda})Q_{j+1}e_1 + g_j(\sqrt{\lambda})Q_{j+1}e_2$.

  Now let us denote by α_j, $\tilde{\alpha_j}$, β_j and $\tilde{\beta_j}$ the terms of the first and second columns of the matrix Q_{j+1} i.e.

$$\begin{cases} \alpha_j = \frac{1}{4}\Big(\frac{c_j}{a_j^{7/4}} - \frac{s_j}{a_j^{7/4}}\Big) \\ \tilde{\alpha_j} = -\frac{1}{4}\frac{c_j}{a_j^3} \\ \beta_j = \frac{1}{4}\frac{2s_j}{a_j^{1/2}} \\ \tilde{\beta_j} = \frac{1}{4}\Big(\frac{c_j}{a_j^{7/4}} - \frac{s_j}{a_j^{7/4}}\Big). \end{cases}$$

It follows

$$Q_{j+1}\left(Q_jQ_{j-1}\cdots Q_1e_1\right) = \left(\alpha_{j+1}f_j + \tilde{\alpha}_{j+1}g_j\right)e_1 + \left(\beta_{j+1}f_j + \tilde{\beta}_{j+1}g_j\right)e_2.$$

Denote by f_{j+1} and g_{j+1} the expressions:

$$f_{j+1} = \alpha_{j+1}f_j + \tilde{\alpha}_{j+1}g_j \text{ and } g_{j+1} = \beta_{j+1}f_j + \tilde{\beta}_{j+1}g_j.$$

After some computation, the Wronskian of f_{j+1} and g_{j+1} is:

$$\begin{aligned} W(f_{j+1}, g_{j+1}) &= \frac{1}{16a_j^{19/4}}\left[2a_j^{5/2}b_jf_j^2 + b_jg_j^2 - 2b_ja_j^{5/4}f_jg_j + a_j^{5/4}W(f_j,g_j)\right] \\ &= \frac{1}{16a_j^{19/4}}b_j\left[a_j^{5/2}f_j^2 + (a_j^{5/4}f_j - g_j)^2 + a_j^{5/4}W(f_j,g_j)\right] > 0 \end{aligned}$$

since $W(f_j, g_j)$ is assumed to be strictly positive.

We have proved, by induction, that, for any

$$j \in \mathbb{N},\ Q_jQ_{j-1}\cdots Q_1e_1 = f_j(\sqrt{\lambda})e_1 + g_j(\sqrt{\lambda})e_2$$

with the Wronskian of f_j and g_j strictly positive for any real x.

- The second step is to compute $f_\infty\left(\sqrt{\lambda}\right) = \left(e_1^tH_{12}^N\right)\left(Q_{N-1}Q_{N-2}\cdots Q_1e_1\right)$ using the first step. It holds:

$$\begin{aligned} \left(e_1^tH_{12}^N\right)\left(Q_{N-1}Q_{N-2}\cdots Q_1e_1\right) &= \left(e_1^tH_{12}^N\right)\left(f_{N-1}\,e_1 + g_{N-1}\,e_2\right) \\ &= \left(f_{N-1}\,e_1^tH_{12}^Ne_1 + g_{N-1}\,e_1^tH_{12}^Ne_2\right) \\ &= f_Nc_N + g_Ns_N \end{aligned}$$

with the Wronskian of f_{N-1} and g_{N-1} strictly positive for any real x, $f_N = \frac{g_{N-1}}{4a_N^{7/4}}$ and $g_N = \frac{f_{N-1}}{2a_N^{1/2}} - \frac{g_{N-1}}{4a_N^{7/4}}$.

Now the linearity of the Wronskian implies that

$$W(f_N, g_N) = -\frac{1}{4a_N^{7/4}}\frac{1}{2a_N^{1/2}}W(f_{N-1}, g_{N-1}) = d < 0\,. \tag{19}$$

- The third step is to compute the derivative of $f_\infty(x)$.

$$f'_\infty(x) = \cos(b_N x)[f'_N(x) + b_N g_N(x)] + \sin(b_N x)[g'_N(x) - b_N f_N(x)]\,.$$

We deduce that for all $x \in \mathbb{R}$, $\Delta(x) = (f_\infty(x))^2 + (f'_\infty(x))^2$ has the following form:

$$\Delta(x) = (\cos(b_N x)\sin(b_N x))M(x)\begin{pmatrix} \cos(b_N x) \\ \sin(b_N x) \end{pmatrix}, \tag{20}$$

where the matrix $M(x)$ is symmetric, positive and given by

$$M(x) = \begin{pmatrix} M_{11}(x) & M_{12}(x) \\ M_{21}(x) & M_{22}(x) \end{pmatrix}$$

and

$$\begin{cases} M_{11}(x) = f_N(x)^2 + b_N^2 g_N(x)^2 + 2b_N g_N(x) f'_N(x) + f'_N(x)^2 \\ M_{12}(x) = (1-b_N^2) f_N(x) g_N(x) - b_N(f_N(x) f'_N(x) - g_N(x) g'_N(x)) + f'_N(x) g'_N(x) \\ M_{21}(x) = M_{12}(x) \\ M_{22}(x) = b_N^2 f_N(x)^2 + g_N(x)^2 - 2b_N f_N(x) g'_N(x) + g_N{}'(x)^2\,. \end{cases}$$

Let $\lambda_{\min}(x), \lambda_{\max}(x)$ be the two eigenvalues of $M(x)$ such that $0 \le \lambda_{\min}(x) \le \lambda_{\max}(x)$. After some computation we find

$$\begin{aligned} \lambda_{\min}(x)\lambda_{\max}(x) &= \det(M(x)) \\ &= b_N^2(f_N(x)^2 + g_N(x)^2)^2 - 2b_N(f_N(x)^2 + g_N(x)^2) W(f_N, g_N)(x) \\ &\quad + W(f_N, g_N)(x)^2. \\ &= \left(W(f_N, g_N)(x) + b_N(f_N(x)^2 + g_N(x)^2)\right)^2. \end{aligned}$$

Consequently with (19),

$$\forall x \in \mathbb{R},\ \det(M(x)) = \lambda_{\min}(x)\lambda_{\max}(x) \ge W(f_N, g_N)(x)^2 \ge d^2. \tag{21}$$

On the other hand, since f_N and g_N are trigonometric polynomials, the trace of $M(x)$ is bounded on $\mathbb{R}$. Thus, there exists $d' > 0$ such that

$$\forall x \in \mathbb{R},\ 0 \le \operatorname{tr}(M(x)) = \lambda_{\min}(x) + \lambda_{\max}(x) \le d'^2. \tag{22}$$

From (21) and (22) we deduce that $\lambda_{\min}(x) \ge \left(\frac{d}{d'}\right)^2 > 0$. Therefore from (20) we get

$$\forall x \in \mathbb{R},\ \Delta(x) \ge \left(\frac{d}{d'}\right)^2 > 0.$$

That means that if x_0 is a root of f_∞ then $|f'_\infty(x_0)| \ge \frac{d}{d'} > 0$. $\square$

All the required properties are now proved to state the main result of this section:

Theorem 13 (The spectral gap).

Let λ_k^2, $k \in \mathbb{N}^*$, $(\lambda_k > 0)$ be the (strictly) monotone increasing sequence of eigenvalues of Problem (EP) given at the beginning of Section 4 then

$$\lim_{k \to +\infty} (\lambda_{k+1}^2 - \lambda_k^2) = +\infty. \tag{23}$$

Proof. We first recall that the roots of f_∞ are simple (cf. Lemma 12). On the other hand, since f_∞ and all its derivatives are trigonometric polynomials, they are all bounded on $\mathbb{R}$. Then

$$\forall x \in \mathbb{R},\ |f'_\infty(x+h) - f'_\infty(x)| = |f''_\infty(x+\theta h)| \cdot |h| \le \|f''_\infty\|_\infty \cdot |h| \tag{24}$$

and it follows that f'_∞ is uniformly continuous on $\mathbb{R}$.

Thus, there exists $h_0 > 0$ such that, for any x_0 satisfying $f_\infty(x_0) = 0$

$$|x - x_0| \le h_0 \Rightarrow |f'_\infty(x)| \ge \frac{d}{2}.$$

Due to Rolle's Theorem, this property implies that x_0 is the unique root of f_∞ in the interval $[x_0 - h_0, x_0 + h_0]$, which also means that the minimal distance between two consecutive roots of f_∞ is h_0.

Multiplying the characteristic equation (14) by $\left(\frac{1}{C \cdot \lambda^{C'} \cdot e^{C''\sqrt{\lambda}}}\right)$ where the constants C, C' and C'' are defined in the proof of Lemma 11, implies

$$\widetilde{f}(\sqrt{\lambda}) = f_\infty(\sqrt{\lambda}) + g(\sqrt{\lambda}) = 0,$$

where the function g is analytical on $\mathbb{R}_+^*$. There exists a constant $C > 0$ such that for all $x \ge 1, g(x) \le \dfrac{C}{x}$ and $\dfrac{dg}{dx}(x) \le \dfrac{C}{x}$. Consequently, using (24) and the relation $\widetilde{f} = f_\infty + g$ and proceeding as for f_∞, we can see that there exists $X_0 \ge 1$ such that $\widetilde{f}'$ is uniformly continuous on $[X_0, +\infty)$. As previously we deduce that the minimal distance between two consecutive nonnegative roots of $\widetilde{f}$ is a constant $h_0' > 0$. The spectral gap is a direct consequence of this property. ☐

5. A useful estimate for controllability

The aim of this section is to prove the following estimate for any eigenfunction ϕ associated to the eigenvalue λ^2 (we still talk about the eigenvalue problem denoted by (EP) in Section 4): there exist constants K_1 and K_2 such that for large values of λ

$$K_1 \cdot \|\phi\|_H^2 \cdot \lambda \le |\phi_1'(0)|^2 \le K_2 \cdot \|\phi\|_H^2 \cdot \lambda \tag{25}$$

with the norm $\|\cdot\|_H$ defined by (1).

5.1 First estimate: observability

Let us begin with the estimate

$$K_1 \cdot \|\phi\|_H^2 \cdot \lambda \leq |\phi_1'(0)|^2 \,. \tag{26}$$

Notation. Consider the functions $h_i(a_j, b_j, \lambda, x)$ for $i \in \{1; 2; 3; 4\}$ and $x \in [0; l_j]$ denoted $h_i(x)$ for the sake of simplicity:

$$\begin{cases} h_1(x) = \cos(a_j^{1/4}\sqrt{\lambda}x) \\ h_2(x) = \sin(a_j^{1/4}\sqrt{\lambda}x) \\ h_3(x) = \exp(-b_j\sqrt{\lambda})\exp(a_j^{1/4}\sqrt{\lambda}x) \\ h_4(x) = \exp(-a_j^{1/4}\sqrt{\lambda}x) \end{cases}$$

$G(a_j, b_j)$ is the 4×4 Gram matrix defined by $(G(a_j, b_j))_{i,k} = \displaystyle\int_0^{l_j} h_i(x)h_k(x)dx$.

$$D(a,b) = \frac{1}{4}\begin{pmatrix} 2 & -\frac{2}{a^{3/2}} & 0 & 0 \\ 0 & 0 & \frac{2}{a^{1/4}} & -\frac{2}{a^{7/4}} \\ \exp(b\sqrt{\lambda}) & \frac{\exp(b\sqrt{\lambda})}{a^{3/2}} & \frac{\exp(b\sqrt{\lambda})}{a^{1/4}} & \frac{\exp(b\sqrt{\lambda})}{a^{7/4}} \\ 1 & \frac{1}{a^{3/2}} & -\frac{1}{a^{1/4}} & -\frac{1}{a^{7/4}} \end{pmatrix}$$

$$E(a,b) = \begin{pmatrix} \cos(b\sqrt{\lambda}) & \sin(b\sqrt{\lambda}) & 1 & \exp(-b\sqrt{\lambda}) \\ -a^{3/2}\cos(b\sqrt{\lambda}) & -a^{3/2}\sin(b\sqrt{\lambda}) & a^{3/2} & a^{3/2}\exp(-b\sqrt{\lambda}) \\ -a^{1/4}\sin(b\sqrt{\lambda}) & -a^{1/4}\cos(b\sqrt{\lambda}) & a^{1/4} & -a^{1/4}\exp(-b\sqrt{\lambda}) \\ a^{7/4}\sin(b\sqrt{\lambda}) & -a^{7/4}\cos(b\sqrt{\lambda}) & a^{7/4} & -a^{7/4}\exp(-b\sqrt{\lambda}) \end{pmatrix}.$$

Remark 14 It will be seen in the following that the choice of the basis h_i is crucial for the asymptotic behaviour of the eigenfunctions to be studied since the expressions are very complicated especially for large values of N. In particular the exponential factor in h_3 has an important role since its presence keeps the exponential terms from being disseminated in the different matrices which would not allow an easy estimation of the involved quantities as λ tends to infinity.

Lemma 15

Any eigenfunction ϕ associated to the eigenvalue λ^2 for the eigenvalue problem (EP) given at the beginning of Section 4 may be uniquely written as a linear combination of the (h_i)'s. Denote by $(C_j)_i$ the coefficients of the decomposition of ϕ_j in the basis $(h_i)_{i\in\{1;2;3;4\}}$ i.e. $\phi_j(x) = \sum_{i=1}^{4} (C_j)_i \, h_i(x)$ for $j \in \{1, \cdots, N\}$ and $x \in [0, l_j]$. Then

$$C_j = D(a_j, b_j)V_j(0), \; V_j(l_j) = E(a_j, b_j)C_j \text{ and } A(a,b) = E(a,b)D(a,b) \tag{27}$$

with V_j and $A(a,b)$ defined in Section 4.1.

(V_j being computed for the $j-th$ component of the particular eigenfunction ϕ).

There exists a positive constant C (by constant we mean independent of λ) such that

$$\|\phi\|_H^2 = \sum_{j=1}^{N} \int_0^{l_j} \phi_j(x)^2 dx + \sum_{i=1}^{N-1} M_i \phi_i(l_i)^2 \leq C \max_{j \in \{1 \cdots N\}} \left(C_j^t C_j + (e_1^t V_j(l_j))^2 \right)$$

where $e_1^t = (1, 0, 0, 0)$.

Proof. Proving that the h_i's are linearly independent is a classical computation. (27) is proved by calculation.

By definition of (1)

$$\|\phi\|_H^2 = \sum_{j=1}^{N} \int_0^{l_j} \phi_j(x)^2 dx + \sum_{i=1}^{N-1} M_i \phi_i(l_i)^2 = \sum_{j=1}^{N} C_j^t G(a_j, b_j) C_j + \sum_{i=1}^{N-1} M_i \phi_i(l_i)^2.$$

Now, after calculation, the matrix $G(a, b)$ is:

$$\frac{1}{4a^{1/4}\sqrt{\lambda}} \begin{pmatrix} 2b\sqrt{\lambda} + \sin(2b\sqrt{\lambda}) & 2(1-c^2) & 2(c+s-e^{-b\sqrt{\lambda}}) & 2(1-e^{-b\sqrt{\lambda}}(c-s)) \\ 2(1-c^2) & 2b\sqrt{\lambda} - \sin(2b\sqrt{\lambda}) & 2(e^{-b\sqrt{\lambda}} - c + s) & 2(1-e^{-b\sqrt{\lambda}}(c+s)) \\ 2(c+s-e^{-b\sqrt{\lambda}}) & 2(-c+s-e^{-b\sqrt{\lambda}}) & 2(1-e^{-2\sqrt{\lambda}}) & 4b\sqrt{\lambda}e^{-b\sqrt{\lambda}}) \\ 2(1-e^{-b\sqrt{\lambda}}(c-s)) & 2(1-e^{-b\sqrt{\lambda}}(c+s)) & 4b\sqrt{\lambda}e^{-b\sqrt{\lambda}}) & 2(1-e^{-2b\sqrt{\lambda}}) \end{pmatrix}$$

with the notation $c = \cos(b\sqrt{\lambda})$, $s = \sin(b\sqrt{\lambda})$.

Note that all its terms are bounded with respect to λ. Moreover, by definition of $V_j(0)$, $\phi_j(l_j)$ is the first component of $V_j(0)$. The estimate of $\|\phi\|_H^2$ follows. □

Lemma 16 (Estimate of $\phi_1'(0)$).

Let $M(\lambda)$ the 4×4 matrix defined by (12) in Section 4.1. Denote by $\alpha(\lambda)$ (respectively $\beta(\lambda)$) the third (resp. fourth) term of the first line of $M(\lambda)$ i.e.

$$\begin{cases} \alpha(\lambda) = e_1^t M(\lambda) e_3 \\ \beta(\lambda) = e_1^t M(\lambda) e_4 \end{cases}$$

with $e_1 = (1, 0, 0, 0)$, $e_3 = (0, 0, 1, 0)$, $e_4 = (0, 0, 0, 1)$.

Then the eigenfunction ϕ of Problem (EP) associated to the eigenvalue λ^2 can be chosen such that $\phi_1'(0) = \sqrt{\lambda} \cdot \beta(\lambda)$ and the asymptotic behaviour of $\alpha(\lambda)$ and $\beta(\lambda)$ is given by:

$$\begin{cases} \alpha(\lambda) = \left[\dfrac{\left(\prod_{i=1}^{N-1} M_i\right)}{4^N \left(\prod_{j=2}^{N} a_j\right)^{7/4} a_1^{1/4}} \right] e^{B\sqrt{\lambda}} \left(\sqrt{\lambda}\right)^{N-1} + o\left(e^{B\sqrt{\lambda}} \left(\sqrt{\lambda}\right)^{N-1} \right) \\ \beta(\lambda) = \left[\dfrac{\left(\prod_{i=1}^{N-1} M_i\right)}{4^N \left(\prod_{j=1}^{N} a_j\right)^{7/4}} \right] e^{B\sqrt{\lambda}} \left(\sqrt{\lambda}\right)^{N-1} + o\left(e^{B\sqrt{\lambda}} \left(\sqrt{\lambda}\right)^{N-1} \right) \end{cases}$$

with $B := \sum_{j=1}^{N} b_j$.

Proof. Any eigenfunction ϕ associated to the eigenvalue λ^2 satisfies (6) and (7). In particular $\phi_1(0) = \phi_1''(0) = 0$ so the first two components of the vector $V_1(0)$ associated to ϕ (defined in Section 4.1) vanish: $V_1(0)$ is of the form $(0, 0, (V_1(0))_3, (V_1(0))_4)^t$. Moreover (6) also implies $\phi_N(l_N) = 0$ so the first component of $V_N(l_N)$ vanishes.

Now, due to Lemma 5, $V_N(l_N) = M(\lambda)V_1(0)$. Thus $\alpha(\lambda)(V_1(0))_3 + \beta(\lambda)(V_1(0))_4 = 0$. $(V_1(0))_3 = \beta(\lambda)$ and $(V_1(0))_4 = -\alpha(\lambda)$ is a solution of this equation which means that the eigenfunction ϕ of Problem (EP) associated to the eigenvalue λ^2 can be chosen such that $\dfrac{\phi_1'(0)}{\sqrt{\lambda}} = \beta(\lambda)$ (such an eigenfunction is not normalized).

The second part of the proof contains the estimate of some terms of the matrix $M(\lambda)$. Recall that

$$M(\lambda) = A_N T_{N-1} A_{N-1} ... A_2 T_1 A_1.$$

There are two types of matrices in that product.

The asymptotic behaviour of T_j is given by $T_j = \sqrt{\lambda} M_j S + o(\sqrt{\lambda})$ with

$$S := \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

That of $A_j = A(a_j, b_j)$ is given by

$$A_j = \exp(b_j\sqrt{\lambda}) B_1(a_j, b_j) + o(\exp(b_j\sqrt{\lambda}))$$

with $B_1(a_j, b_j)$ defined by the decomposition

$$A(a, b) = \sum_{\epsilon \in \{-1;0;1\}} \exp(\epsilon b\sqrt{\lambda}) B_\epsilon(a, b).$$

Note that $B_1(a, b)$ and $B_{-1}(a, b)$ only depend on a.

An iterative calculation leads to:

$$SB_1(a_{N-1})S \cdots SB_1(a_1) := \frac{1}{4^{N-1}\Big(\prod_{j=2}^{N-1} a_j\Big)^{7/4}} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & a_1^{3/2} & a_1^{1/4} & a_1^{7/4} \end{pmatrix}$$

with $\Big(\prod_{j=2}^{N-1} a_j\Big)^{7/4} = 1$ for $N = 2$. Combining the above estimates and using (12) lead to the desired result. $\square$

The aim is still the estimation of $\|\phi\|_H$ which requires, due to Lemma 16, the estimation of C_j.

Lemma 17 (Estimate of C_j).

Let C_j be the vector already defined by (27) with a fixed $j \in \{1, \cdots, N\}$, the *vector $\vec{b}$ be defined by $\vec{b} = (b_1, \cdots, b_N)$ and denote by $\vec{u}\cdot\vec{v} := \sum_{i=1}^{N} u_i v_i$, then there exist vectors $W_{\vec{\epsilon}}(\lambda)$ such that C_j is of the form*

$$C_j := \sum_{\vec{\epsilon}\in\{-2;-1;0;1\}^N} e^{\vec{b}\cdot\vec{\epsilon}\sqrt{\lambda}} W_{\vec{\epsilon}}(\lambda) \tag{28}$$

and all the terms of $W_{\vec{\epsilon}}(\lambda)$ are dominated by the function $\lambda \mapsto e^{B\sqrt{\lambda}}$ asymptotically (with $B := \sum_{j=1}^{N} b_j$).

More precisely the terms of $W_{\vec{\epsilon}}(\lambda)$ only contain expressions of the form $\lambda^{p/2}$ with p an integer and terms of the form $\cos(b_j\sqrt{\lambda})$ and $\sin(b_j\sqrt{\lambda})$ with $j \in \{1, ..., N\}$.

Proof.

First Part. For a fixed $j \in \{1, \cdots, N\}$, we start with isolating the terms containing $e^{b_j\sqrt{\lambda}}$ in the involved matrices.

$$\begin{cases} D(a_j, b_j) = e^{b_j\sqrt{\lambda}} D^+(a_j) + D^r(a_j, b_j) \\ A_j = A(a_j, b_j) = e^{b_j\sqrt{\lambda}} B^+(a_j) + B^r(a_j, b_j). \end{cases}$$

The decomposition of A_j is the same one as in the proof of Lemma 16 i.e. the matrix called B^+ in that proof is B_1. The exponent r is chosen for the rest which does not contain $e^{b_j\sqrt{\lambda}}$.

Since $M(\lambda) = A_N T_{N-1} \cdots T_j A_j T_{j-1} \cdots A_2 T_1 A_1$ (cf. (12)) and since only A_j contains $e^{b_j\sqrt{\lambda}}$, it holds:

$$\begin{aligned} M(\lambda) &= A_N T_{N-1} \cdots T_j \left(e^{b_j\sqrt{\lambda}} B^+(a_j, b_j) + B^r(a_j, b_j)\right) T_{j-1} \cdots A_2 T_1 A_1 \\ &= e^{b_j\sqrt{\lambda}} A_N T_{N-1} \cdots T_j B^+(a_j, b_j) T_{j-1} \cdots A_2 T_1 A_1 \\ &\quad + A_N T_{N-1} \cdots T_j B^r(a_j, b_j) T_{j-1} \cdots A_2 T_1 A_1 \\ &=: e^{b_j\sqrt{\lambda}} M^+(\lambda) + M^r(\lambda). \end{aligned}$$

The third and fourth terms of the first line of the matrix $M(\lambda)$ denoted by α and β in Lemma 16 can be decomposed as follows:

$$\begin{cases} \alpha(\lambda) = e_1^t M(\lambda) e_3 = e^{b_j\sqrt{\lambda}} e_1^t M^+(\lambda) e_3 + e_1^t M^r(\lambda) e_3 = e^{b_j\sqrt{\lambda}} \alpha^+(\lambda) + \alpha^r(\lambda) \\ \beta(\lambda) = e_1^t M(\lambda) e_4 = e^{b_j\sqrt{\lambda}} \beta^+(\lambda) + \beta^r(\lambda). \end{cases}$$

Thus the vector $V_1(0)$ is decomposed as well:

$$V_1(0) = (0, 0, \beta(\lambda), -\alpha(\lambda))^t = e^{b_j\sqrt{\lambda}} V_1^+(0) + V_1^r(0)$$

with $V_1^+(0) = (0, 0, \beta^+(\lambda), -\alpha^+(\lambda))^t$. Then $C_j = D(a_j, b_j) T_{j-1} A_{j-1} \cdots T_1 A_1 V_1(0)$ may be written as:

$$C_j = e^{2b_j\sqrt{\lambda}} C_j^{++} + e^{b_j\sqrt{\lambda}} C_j^{+} + C_j^r \text{ with } C_j^{++} := D^+(a_j) T_{j-1} A_{j-1} \cdots T_1 A_1 V_1^+(0) \tag{29}$$

where neither C_j^{++}, nor C_j^{+}, nor C_j^{r} contains $e^{b_j\sqrt{\lambda}}$. The vanishing of C_j^{++} remains to be proved in order to establish (28).

Second Part. For a fixed $j \in \{1, \cdots, N\}$, let us prove that $C_j^{++} = 0$ with C_j^{++} defined by (29).

Recall that $M(\lambda) = e^{b_j\sqrt{\lambda}}M^+(\lambda) + M^r(\lambda)$ with

$$M^+(\lambda) = A_N T_{N-1} \cdots T_j B^+(a_j) T_{j-1} \cdots A_2 T_1 A_1 \,.$$

The matrices T_i for $i \in \{1, \cdots, j-1\}$ defined in Section 4.1 are all clearly invertible. The matrices A_i for $i \in \{1, \cdots, j-1\}$ defined in the same section have the same property. It is less clear but their determinant is also equal to 1 (calculation).

The matrix $B^+(a_j)$ is defined as follows:

$$B^+(a_j) = \frac{1}{4}\begin{pmatrix} 1 & \dfrac{1}{a_j^{3/2}} & \dfrac{1}{a_j^{1/4}} & \dfrac{1}{a_j^{7/4}} \\ a_j^{3/2} & 1 & a_j^{5/4} & \dfrac{1}{a_j^{1/4}} \\ a_j^{1/4} & \dfrac{1}{a_j^{5/4}} & 1 & \dfrac{1}{a_j^{3/2}} \\ a_j^{7/4} & a_j^{1/4} & a_j^{3/2} & 1 \end{pmatrix}.$$

Note that the columns of $B^+(a_j)$ are all proportional to the first one so the rank of $B^+(a_j)$ is 1. Thus the rank of $M^+(\lambda)$ is also 1 which means in particular that all its lines are proportional to the first one.

Now the third (respectively fourth) term of the first line of $M^+(\lambda)$ is, by definition, $\alpha^+(\lambda)$ (resp. $\beta^+(\lambda)$) and $V_1^+(0) = (0, 0, \beta^+(\lambda), -\alpha^+(\lambda))^t$. So the first term of the product $M^+(\lambda)V_1^+(0)$ is 0. And since the other lines of $M^+(\lambda)$ are proportional to the first one, the other terms also vanish i.e. $M^+(\lambda)V_1^+(0) = 0$.

It is equivalent to $A_N T_{N-1} \cdots T_j B^+(a_j) T_{j-1} \cdots A_2 T_1 A_1 V_1^+(0) = 0$ and, since $(A_N T_{N-1} \cdots T_j)$ is invertible, it implies: $T_{j-1} \cdots A_2 T_1 A_1 V_1^+(0) \in Ker(B^+(a_j))$.

The matrix $D^+(a_j, b_j)$ is defined as follows:

$$D^+(a_j) = \frac{1}{4}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & \dfrac{1}{a_j^{3/2}} & \dfrac{1}{a_j^{1/4}} & \dfrac{1}{a_j^{7/4}} \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

It clearly holds $\mathrm{Ker}(B^+(a_j)) = \mathrm{Ker}(D^+(a_j))$.

Thus $C_j^{++} := D^+(a_j) T_{j-1} A_{j-1} \cdots A_2 T_1 A_1 V_1^+(0) = 0$. □

Lemma 18 (Estimate of $V_j(l_j)$).

Let $j \in \{1, \cdots, N\}$ and let the vector V_j be defined as in Section 4.1. For any $K > 0$, there exists a positive constant C (independent of λ) such that, if $\lambda > K$:

$$|e_1^t V_j(l_j)| \leq C e^{B\sqrt{\lambda}} (\sqrt{\lambda})^{N-2} \tag{30}$$

with $e_1^t = (1,0,0,0)$. *Moreover, for* $j \in \{1, \cdots, N\}$ *and* $i \in \{2;3;4\}$

$$|e_i^t V_j(l_j)| \leq C e^{B\sqrt{\lambda}} (\sqrt{\lambda})^{N-1} \tag{31}$$

with $e_1^t = (1,0,0,0)$, $e_2^t = (0,1,0,0)$, $e_3^t = (0,0,1,0)$, $e_4^t = (0,0,0,1)$ *and* $B := \sum_{j=1}^N b_j$.

Note that the constants C are not identical nor equal to that of Lemma 15 but we will always call the constants C. All of them are independent of λ but depend on the material constants given by the a_j's, b_j's...

Proof. Both properties will be proved by the same induction. The basis case is for $j = 1$:

Since $M(\lambda) = A_N T_{N-1} \cdots A_2 T_1 A_1$ (cf. (12)) and $T_1 = Id + \sqrt{\lambda} M_1 S$, with S defined as in the proof of Lemma 16, $M(\lambda)$ can be decomposed in the following way: $M(\lambda) = M^1(\lambda) + M^R(\lambda)$ with $M^1(\lambda) = A_N T_{N-1} \cdots A_2 (\sqrt{\lambda} M_1 S) A_1$. That is to say the fastest growing term in $M^1(\lambda)$ is of the form

$$C\Big(e^{B\sqrt{\lambda}}\big(\sqrt{\lambda}\big)^{N-1}\Big)\Big(\prod_{i=1}^{N-1} M_i\Big).$$

And the fastest growing term in $M^R(\lambda)$ is of the form

$$C\Big(e^{B\sqrt{\lambda}}\big(\sqrt{\lambda}\big)^{N-2}\Big)\Big(\prod_{i=2}^{N-1} M_i\Big).$$

Then the third and fourth terms of the first line of the matrix $M(\lambda)$ denoted by α and β in Lemma 16 can be decomposed as follows:

$$\begin{cases} \alpha(\lambda) = e_1^t M(\lambda) e_3 = \alpha^1(\lambda) + \alpha^R(\lambda) \\ \beta(\lambda) = e_1^t M(\lambda) e_4 = \beta^1(\lambda) + \beta^R(\lambda) \end{cases}$$

with

$$\begin{cases} \alpha^1(\lambda) = e_1^t M^1(\lambda) e_3 \\ \beta^1(\lambda) = e_1^t M^1(\lambda) e_4 \,. \end{cases}$$

Thus the vector $V_1(0)$ is decomposed as well:

$$V_1(0) = (0, 0, \beta(\lambda), -\alpha(\lambda))^t = V_1^1(0) + V_1^R(0)$$

with $V_1^1(0) = (0, 0, \beta^1(\lambda), -\alpha^1(\lambda))^t$.

The end of the proof is similar to the second part of the proof of Lemma 17: $M^1(\lambda) = A_N T_{N-1} \cdots A_2 (\sqrt{\lambda} M_1 S) A_1$ where S is a matrix with rank 1 and the matrices A_i and T_i are invertible. Then the rank of $M^1(\lambda)$ is also 1 which means in particular that all its lines are proportional to the first one.

Now the third (respectively fourth) term of the first line of $M^1(\lambda)$ is, by definition, $\alpha^1(\lambda)$ (resp. $\beta^1(\lambda)$) and $V_1^1(0) = (0, 0, \beta^1(\lambda), -\alpha^1(\lambda))^t$. So the first term of the product $M^1(\lambda) V_1^1(0)$ is 0. And since the other lines of $M^1(\lambda)$ are proportional to the first one, the other terms also vanish i.e. $M^1(\lambda) V_1^1(0) = 0$.

It is equivalent to $A_N T_{N-1} \cdots A_2(\sqrt{\lambda} M_1 S) A_1 V_1^1(0) = 0$ and, since $(A_N T_{N-1} \cdots A_2)$ is invertible, it implies: $S A_1 V_1^1(0) = 0$ which is equivalent to $S V_1^1(l_1) = 0$ (cf. Lemma 5).

Thus $|e_1^t V_1^1(l_1)| = 0$ which gives the desired result (30) for $j = 1$ by definition of V_1^1.

As for the other terms of $V_1(l_1) = A_1 V_1(0)$, they keep the same fastest growing term as the terms of $V_1(0)$ i.e.

$$C\left(e^{B\sqrt{\lambda}}\left(\sqrt{\lambda}\right)^{N-1}\right)\left(\prod_{i=1}^{N-1} M_i\right).$$

Indeed the multiplication by the matrix A_1 which contains exponential terms could a priori change the exponential into $e^{\left(2b_1+\sum_{j=2}^{N} b_j\right)\sqrt{\lambda}}$ but as it was proved for C_j in the proof of Lemma 17, it is not the case.

Let us now establish the induction rule.

Assume that (30) and (31) are true for $j \leq m$. The aim is to prove that they are still true for $j = m + 1$.

$$V_{m+1}(0) = T_m V_m(l_m) = V_m(l_m) + \sqrt{\lambda} M_m S V_m(l_m)$$

(cf. Lemma 5) so the first three terms of $V_{m+1}(0)$ are equal to those of $V_m(l_m)$ and the fourth one is

$$|e_4^t V_{m+1}(0)| = |e_4^t V_m(l_m) + M_m \sqrt{\lambda} e_1^t V_m(l_m)| \leq C e^{B\sqrt{\lambda}} (\sqrt{\lambda})^{N-1}$$

for large values of λ due to (30) and (31). □

Lemma 19 (A more precise estimate of C_j).

Let C_j be the vector already defined by (27) with a fixed $j \in \{1, \cdots, N\}$. For any $K > 0$, there exists a constant C (independent of λ) such that, if $\lambda > K$ and $i \in \{1; 2; 3; 4\}$:

$$|(C_j)_i| \leq C e^{B\sqrt{\lambda}} (\sqrt{\lambda})^{N-1} \tag{32}$$

with $(C_j)_i$ the $i - th$ term of the vector C_j as in Lemma 15 and $B := \sum_{j=1}^{N} b_j$.

Proof. Recall that $C_j = D(a_j, b_j) V_j(0)$ (cf. (27)). We have just proved in the second part of the proof of Lemma 18 that, for any $j \in \{1, \cdots, N\}$, the absolute values of the four terms of $V_{j+1}(0)$ are bounded by $C e^{B\sqrt{\lambda}} (\sqrt{\lambda})^{N-1}$ for large values of λ. It is also clear for $V_1(0) = (0, 0, \beta(\lambda), -\alpha(\lambda))$ due to the estimates of α and β given in Lemma 16.

Now the matrix $D(a_j, b_j)$ contains exponential terms but we proved in the proof of Lemma 17 that they do not affect the fastest growing term of C_j. Hence the result. □

Proposition 20 (First estimate for controllability).

Consider the eigenvalue problem (EP) associated to Problem (P) (given in Section 4). For any eigenfunction ϕ associated to the eigenvalue λ^2 and for any $K > 0$, there exist a constant K_1 such that, if $\lambda > K$:

$$K_1 \cdot \|\phi\|_H^2 \cdot \lambda \leq |\phi_1'(0)|^2 \tag{33}$$

with the norm $\|\cdot\|_H$ defined by (1).

Proof. Due to Lemma 15

$$\|\phi\|_H^2 \leq C \max_{j \in \{1 \cdots N\}} \left(C_j^t C_j + (e_1^t V_j(l_j))^2 \right)$$

(30) and (32) imply $\|\phi\|_H^2 \leq C \left(e^{B\sqrt{\lambda}} (\sqrt{\lambda})^{N-1} \right)^2$.

Now, by definition of V_j (cf. Section 4.1), $|\phi_1'(0)|^2 = |\sqrt{\lambda}((V_1)(0))_3|^2 = \lambda |\beta(\lambda)|^2$. The estimate of $\beta(\lambda)$ for large values of λ given in Lemma 16 gives the desired result. □

5.2 Second estimate: admissibility

Proposition 21 (Second estimate for controllability).

Consider the eigenvalue problem (EP) associated to Problem (P) (given in Section 4). For any eigenfunction ϕ associated to the eigenvalue λ^2 and for any $K > 0$, there exists a constant K_2 such that, if $\lambda > K$:

$$|\phi_1'(0)|^2 \leq K_2 \cdot \|\phi\|_H^2 \cdot \lambda \tag{34}$$

with the norm $\|\cdot\|_H$ defined by (1).

Proof. We established in the proof of Lemma 15

$$\|\phi\|_H^2 = \sum_{j=1}^{N} \int_0^{l_j} \phi_j(x)^2 dx + \sum_{i=1}^{N-1} M_i \phi_i(l_i)^2 = \sum_{j=1}^{N} C_j^t G(a_j, b_j) C_j + \sum_{i=1}^{N-1} M_i \phi_i(l_i)^2$$

with C_j and $G(a_j, b_j)$ defined in the same lemma. Thus $\|\phi\|_H^2 \geq C_1^t G(a_1, b_1) C_1$ and it remains to estimate this expression from below.

Due to (27) it holds $C_1 = D(a_1, b_1) V_1(0)$ and we stated in Lemma 16 that

$$V_1(0) = (0, 0, \beta(\lambda), -\alpha(\lambda)) = (0, 0, \beta, -a_1^{3/2} + o(\beta))$$

as λ and thus β tend to infinity. Then, multiplying by the matrix $D(a_1, b_1)$ given just before Lemma 15, it follows

$$C_1 = \left(0, \frac{\beta}{a_1^{1/4}} - \frac{o(\beta)}{2a_1^{1/4}}, \frac{e^{b_1\sqrt{\lambda}} o(\beta)}{4a_1^{7/4}}, o(\beta) \right).$$

Now we proved in Lemma 19 that, for any $K > 0$, there exists a constant C (independent of λ) such that, if $\lambda > K$ and $i \in \{1; 2; 3; 4\}$, then $|(C_j)_i| \leq C\beta(\lambda)$ with $(C_j)_i$ the $i - th$ term of the vector C_j. So the third term of C_1 grows as fast as β i.e.

$$C_1 = \left(0, \frac{\beta}{a_1^{1/4}} - \frac{o(\beta)}{2a_1^{1/4}}, O(\beta), o(\beta) \right).$$

Looking thoroughly at the terms of the matrix $G(a_1, b_1)$ given in the proof of Lemma 15, we can see that only two terms do not tend to zero as λ tends to infinity which can be written as:

$$G(a_1, b_1) = \begin{pmatrix} \frac{b_1}{2a_1^{1/4}} & 0 & 0 & 0 \\ 0 & \frac{b_1}{2a_1^{1/4}} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + o(1).$$

It follows

$$C_1^t.G(a_1, b_1).C_1 = \frac{b_1}{2a_1^{1/4}} \left(\frac{\beta}{a_1^{1/4}} \right)^2 + o(\beta^2)$$

and since $|\phi_1'(0)|^2 = |\sqrt{\lambda}((V_1)(0))_3|^2 = \lambda |\beta(\lambda)|^2$, the desired estimate follows. □

6. Controllability for a chain of N serially connected beams with interior masses

All the required elements to prove controllability are now available. The following theorem is a generalization of [16, Theorem 7].

Theorem 22 (Controllability).

Let T be strictly positive and consider the initial data (U^0, U^1) in $\mathcal{H}_{1/4}$ (cf. [35, Lemma 5 and Proposition 6]).

Then there exists a control $q(t)$ in $L^2(0, T)$ such that the solution of Problem (PC) given in Section 3.1 satisfies

$$\begin{cases} u_j(x, T) = 0, \ \forall x \in k_j, \ j \in \{1, \dots, N\} \\ u_{j,t}(x, T) = 0, \ \forall x \in k_j, \ j \in \{1, \dots, N\} \end{cases}$$

and

$$\begin{cases} z_i(T) = 0, \ \forall i \in I_{\text{int}} \\ z_{i,t}(T) = 0, \ \forall i \in I_{\text{int}} \,. \end{cases}$$

Proof. Following Castro and Zuazua i.e. applying the Hilbert Uniqueness Method (recalled in [35, Section 4.1]), the control problem is reduced to the obtention of the observability inequality (9) for the uncontrolled problem that is to say for Problem (PC) with $q = 0$. Using the representation of the solution as a Fourier series, it is equivalent to show the existence of the spectral gap defined in Lemma 4 as well as the two estimates proved in ([16, cf. Section 4.1]).

Indeed writing the solution as a Fourier series taking into account the fact that the eigenfunctions ϕ_k are not normalized and $\mathcal{H} = \mathcal{H}_{1/4} := X_{1/4} \times X_{-1/4}$ with X_α defined by [35, Proposition 2] which is recalled in the following for the sake of completeness:

Proposition 23

There exists a sequence of real positive eigenvalues $(\lambda_k^2)_k$ of the operator $\mathcal{A}$ (defined in Section 2.2) such that

$$0 < (\lambda_1)^2 \leq (\lambda_2)^2 \leq \cdots \text{ with } \lim_{k \longrightarrow +\infty} \lambda_k = +\infty .$$

The associated eigenfunctions $\hat{\Phi}_k := (\Phi_k, \Phi_k(E_1), \ldots, \Phi_k(E_{N-1}))$ can be chosen to constitute an orthonormal basis of $H = \Pi_{j=1}^N L^2((0,l_j)) \times R^{N-1}$ endowed with the inner product given by (1).

The domains of the powers of the linear operator $\mathcal{A}$ are given by

$$X_\alpha := D(\mathcal{A}^\alpha) = \left\{ (u,z) \mid \exists (\alpha_k)_k, \ (u,z) = \sum_{k \in \mathbb{N}} \alpha_k \hat{\Phi}_k, \ \|u\|_{X_\alpha}^2 = \sum_{k \in \mathbb{N}} |\alpha_k|^2 (\lambda_k)^{4\alpha} < \infty \right\}$$

for any α in $\mathbb{R}$.

Now combining Theorem 13 and Lemma 4 leads to:

$$\begin{aligned} C_1(T) \cdot \sum_{k \in Z - \{0\}} |\alpha_k|^2 |(\phi_k')(0)|^2 &\leq \int_0^T \Big| a_1 \sum_{k \in Z - \{0\}} \alpha_k e^{i\lambda_k t} (\phi_k')_1(0) \Big|^2 dt \\ &\leq C_2(T) \cdot \sum_{k \in Z - \{0\}} |\alpha_k|^2 |(\phi_k')(0)|^2 \end{aligned}$$

where $(\phi_k)_1$ is the restriction to the beam k_1 of the eigenfunction ϕ_k associated to the eigenvalue λ_k^2.

Then, using Proposition 20 and Proposition 21, it follows:

$$\begin{aligned} \kappa_1 \cdot \sum_{k \in Z - \{0\}} \lambda_k |\alpha_k|^2 \|\phi_k\|_H^2 &\leq \int_0^T \Big| a_1 \sum_{k \in Z - \{0\}} \alpha_k e^{i\lambda_k t} (\phi_k')_1(0) \Big|^2 dt \\ &\leq \kappa_2 \cdot \sum_{k \in Z - \{0\}} \lambda_k |\alpha_k|^2 \|\phi_k\|_H^2 \end{aligned}$$

with $\kappa_1 := C_1(T) K_1$ and $\kappa_2 := C_2(T) K_2$.

At last (9)

$$\begin{aligned} &\Longleftrightarrow \kappa_1 \cdot \|(U^0, U^1)\|_{\mathcal{H}}^2 \leq \int_0^T \Big| a_j \frac{\partial u_j}{\partial \nu_j}(E_{i_0}, t) \Big|^2 dt \leq \kappa_2 \cdot \|(U^0, U^1)\|_{\mathcal{H}}^2 \\ &\Longleftrightarrow \kappa_1 \cdot \sum_{k \in Z - \{0\}} \lambda_k |\alpha_k|^2 \|\phi_k\|_H^2 \leq \int_0^T \Big| a_1 \sum_{k \in Z - \{0\}} \alpha_k e^{i\lambda_k t} (\phi_k')_1(0) \Big|^2 dt \leq \kappa_2 \\ &\quad \cdot \sum_{k \in Z - \{0\}} \lambda_k |\alpha_k|^2 \|\phi_k\|_H^2 . \end{aligned}$$

Hence the result. □

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