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On certain estimates for Marcinkiewicz integrals and extrapolation

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Abstract

We obtain L^p estimates for parametric Marcinkiewicz integrals associated to polynomial mappings and with rough kernels on the unit sphere as well as on the radial direction. These estimates will allow us to use an extrapolation argument to obtain some new and improved results on Marcinkiewicz integrals. Also, such estimates provide us with a unifying approach in dealing with Marcinkiewicz integrals when the kernel function Ω belongs to the class of block spaces $B_q^{(0,\alpha)}(\mathbf{S}^{n-1})$ as well as when Ω belongs to the class $L(\log L)^{\alpha}(\mathbf{S}^{n-1})$. Our conditions on the kernels are known to be the best possible in their respective classes.

1. Introduction

Throughout this paper, let $\mathbf{R}^n$, $n \geq 2$, be the n-dimensional Euclidean space and $\mathbf{S}^{n-1}$ be the unit sphere in $\mathbf{R}^n$ equipped with the normalized Lebesgue surface measure $d\sigma$.

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Also, we let ξ' denote $\xi/|\xi|$ for $\xi \in \mathbf{R}^n\backslash\{0\}$ and p' denote the exponent conjugate to p, that is $1/p + 1/p' = 1$.

Let $\mathcal{P}(y) = (P_1(y), \ldots, P_m(y))$ be a polynomial mapping, where each P_j is a real-valued polynomial on $\mathbf{R}^n$. To $\mathcal{P}$ we associate a parametric Marcinkiewicz integral operator $\mathcal{M}^{\rho}_{\mathcal{P},\Omega,h}$ defined initially for C_0^∞ functions on $\mathbf{R}^m$ by

$$\mathcal{M}^{\rho}_{\mathcal{P},\Omega,h}f(x) = \left(\int_0^\infty \left| t^{-\rho} \int_{|u|\le t} f\left(x - \mathcal{P}\left(u\right)\right) \frac{\Omega(u')}{|u|^{n-\rho}} h(|u|) du \right|^2 \frac{dt}{t} \right)^{1/2}$$

where $\rho = \sigma + i\tau$ ($\sigma, \tau \in \mathbf{R}$ with $\sigma > 0$), h is a measurable function on $\mathbf{R}_+$, and $\Omega \in L^1(\mathbf{S}^{n-1})$ and satisfies

$$\int_{\mathbf{S}^{n-1}} \Omega\left(u\right) d\sigma\left(u\right) = 0. \tag{1.1}$$

If $m = n$ and $\mathcal{P}\left(y\right) \equiv y$, we shall simply denote $\mathcal{M}^{\rho}_{\mathcal{P},\Omega,h}$ by $\mathcal{M}^{\rho}_{\Omega,h}$ and we denote $\mathcal{M}^{\rho}_{\Omega,h}$ by $\mathcal{M}_{\Omega}$ if $h \equiv 1$ and $\rho = 1$.

We point out that the class of operators $\mathcal{M}^{\rho}_{\mathcal{P},\Omega,h}$ is related to the class of homogeneous singular integral operators

$$T_{\mathcal{P},\Omega,h}f(x) = \text{p.v.} \int_{\mathbf{R}^n} f\left(x - \mathcal{P}\left(u\right)\right) \frac{\Omega(u')}{|u|^n} h(|u|) du. \tag{1.2}$$

The class of operators defined by (1.2) belongs to the class of singular Radon transforms and it has been studied by many authors. For more information about the importance and the recent development in the study of this class of operators, we refer the readers to [18, 10, 4, 2, 15], among others. The operators $\mathcal{M}^{\rho}_{\mathcal{P},\Omega,h}$ defined in (1.1) have their roots in the classical Marcinkiewicz integral operator $\mathcal{M}_{\Omega}$. In [16], E.M. Stein introduced the operator $\mathcal{M}_{\Omega}$ and proved that if $\Omega \in \text{Lip}_{\alpha}(\mathbf{S}^{n-1})$ $(0 < \alpha \le 1)$, then $\mathcal{M}_{\Omega}$ is of type (p, p) for $1 < p \le 2$ and of weak type $(1, 1)$. Subsequently, the study of $\mathcal{M}_{\Omega}$ and some of its extensions has attracted the attention of many authors. Readers may consult [20, 7, 1, 3, 2, 6], among a large number of references for their development and applications. Before stating some known results relevant to our current study, we need to recall and introduce some definitions.

For $\gamma > 0$, let $\Delta_{\gamma}(\mathbf{R}_+)$ denote the collection of all measurable functions $h : [0, \infty) \longrightarrow \mathbf{C}$ satisfying

$$\|h\|_{\Delta_{\gamma}} = \sup_{k\in\mathbf{Z}} \left(\int_{2^k}^{2^{k+1}} |h(t)|^{\gamma} dt/t \right)^{1/\gamma} < \infty$$

and $\mathcal{L}_{\gamma}(\mathbf{R}_+)$ denote the collection of all measurable functions $h : [0, \infty) \longrightarrow \mathbf{C}$ satisfying

$$L_{\gamma}(h) = \sup_{k\in\mathbf{Z}} \left(\int_{2^k}^{2^{k+1}} |h(t)| \left(\log(2 + |h(t)|) \right)^{\gamma} dt/t \right) < \infty.$$

Also, we let $\mathcal{N}_{\gamma}(\mathbf{R}_+)$ denote the class of all measurable functions h on $\mathbf{R}_+$ such that

$$N_{\gamma}(h) = \sum_{m=1} m^{\gamma} 2^m d_m(h) < \infty,$$

where $d_m(h) = \sup_{k\in\mathbf{Z}} 2^{-k} |E(k,m)|$ with

$$E(k,m) = \{t \in (2^k, 2^{k+1}] : 2^{m-1} < |h(t)| \le 2^m\}$$

for $m \ge 2$ and $E(k,1) = \{t \in (2^k, 2^{k+1}] : |h(t)| \le 2\}$.

Remark 1 It is easy to verify that the following inclusion relations hold and are proper:

(1) $\Delta_{\gamma_2}(\mathbf{R}_+) \subset \Delta_{\gamma_1}(\mathbf{R}_+)$ for $1 \le \gamma_1 < \gamma_2$;
(2) $\mathcal{N}_{\gamma_2}(\mathbf{R}_+) \subset \mathcal{N}_{\gamma_1}(\mathbf{R}_+)$ for $\gamma_1 < \gamma_2$;
(3) $\mathcal{L}_{\gamma_2}(\mathbf{R}_+) \subset \mathcal{L}_{\gamma_1}(\mathbf{R}_+)$ for $\gamma_1 < \gamma_2$;
(4) $\Delta_{\gamma}(\mathbf{R}_+) \subset \mathcal{N}_{\alpha}(\mathbf{R}_+) \subset \mathcal{L}_{\alpha}(\mathbf{R}_+)$ for any $\gamma \ge 1, \alpha > 0$;
(5) For a given $\alpha > 1$, $\mathcal{L}_{\gamma+\alpha}(\mathbf{R}_+) \subset \mathcal{L}_{\gamma}(\mathbf{R}_+)$ for any $\gamma > 0$;
(6) $L(\log L)^{\gamma}(\mathbf{R}_+, dt/t) \subset \mathcal{N}_{\gamma}(\mathbf{R}_+)$ for all $\gamma > 0$, where $L(\log L)^{\gamma}(\mathbf{R}_+, dt/t)$ is the class of all measurable functions h on $\mathbf{R}_+$ which satisfy

$$\int_{\mathbf{R}_+} |h(t)| \left(\log(2 + |h(t)|)\right)^{\gamma} dt/t < \infty.$$

The class $L(\log L)^{\alpha}(\mathbf{S}^{n-1})$ (for $\alpha > 0$) denotes the class of all measurable functions Ω on $\mathbf{S}^{n-1}$ which satisfy

$$\|\Omega\|_{L(\log L)^{\alpha}(\mathbf{S}^{n-1})} = \int_{\mathbf{S}^{n-1}} |\Omega(y)| \log^{\alpha}(2 + |\Omega(y)|) d\sigma(y) < \infty.$$

Now we recall the definition of the block space $B_q^{(0,\upsilon)}(\mathbf{S}^{n-1})$. This space was introduced by Jiang and Lu (see [13]) in their study of the mapping properties of homogeneous singular integral operators and it is defined as follows: A $q-$block on $\mathbf{S}^{n-1}$ is an L^q $(1 < q \le \infty)$ function $b(x)$ that satisfies (i) $\mathrm{supp}(b) \subset I$; (ii) $\|b\|_{L^q} \le |I|^{-1/q'}$, where $|I| = \sigma(I)$, and $I = B(x_0', \theta_0) = \{x' \in \mathbf{S}^{n-1} : |x' - x_0'| < \theta_0\}$ is a cap on $\mathbf{S}^{n-1}$ for some $x_0' \in \mathbf{S}^{n-1}$ and $\theta_0 \in (0,1]$. The block space $B_q^{(0,\upsilon)}(\mathbf{S}^{n-1})$ is defined by

$$B_q^{(0,\upsilon)}(\mathbf{S}^{n-1}) = \Big\{\Omega \in L^1(\mathbf{S}^{n-1}) : \Omega = \sum_{\mu=1}^{\infty} \lambda_{\mu} b_{\mu}, M_q^{(0,\upsilon)}(\{\lambda_{\mu}\}) < \infty\Big\},$$

where each λ_{μ} is a complex number; each b_{μ} is a q-block supported on a cap I_{μ} on $\mathbf{S}^{n-1}$, $\upsilon > -1$ and

$$M_q^{(0,\upsilon)}(\{\lambda_{\mu}\}) = \sum_{\mu=1}^{\infty} |\lambda_{\mu}| \Big\{1 + \log^{(\upsilon+1)}\left(|I_{\mu}|^{-1}\right)\Big\}. \tag{1.3}$$

Let $\|\Omega\|_{B_q^{(0,\upsilon)}(\mathbf{S}^{n-1})} = N_q^{(0,\upsilon)}(\Omega) = \inf\{M_q^{(0,\upsilon)}(\{\lambda_{\mu}\}) : \Omega = \sum_{\mu=1}^{\infty} \lambda_{\mu} b_{\mu}$ and each b_{μ} is a q-block function supported on a cap I_{μ} on $\mathbf{S}^{n-1}\}$. Then $\|\cdot\|_{B_q^{(0,\upsilon)}(\mathbf{S}^{n-1})}$ is a norm on the space $B_q^{(0,\upsilon)}(\mathbf{S}^{n-1})$ and $(B_q^{(0,\upsilon)}(\mathbf{S}^{n-1}), \|\cdot\|_{B_q^{(0,\upsilon)}(\mathbf{S}^{n-1})})$ is a Banach space.

Remark 2 For any $q>1$ and $0<\upsilon\le 1$, the following inclusions hold and are proper:

$$\mathrm{Lip}_\upsilon(\mathbf{S}^{n-1})\subset L^q(\mathbf{S}^{n-1})\subset L(\log L)(\mathbf{S}^{n-1})\subset H^1(\mathbf{S}^{n-1})\subset L^1(\mathbf{S}^{n-1})\,, \tag{1.4}$$

$$\bigcup_{r>1} L^r(\mathbf{S}^{n-1})\subset B_q^{(0,\upsilon)}(\mathbf{S}^{n-1})\ \text{ for any } -1<\upsilon \text{ and } q>1\,, \tag{1.5}$$

$$L(\log L)^\beta(\mathbf{S}^{n-1})\subset L(\log L)^\alpha(\mathbf{S}^{n-1})\ \text{ if } 0<\alpha<\beta\,, \tag{1.6}$$

$$L(\log L)^\alpha(\mathbf{S}^{n-1})\subset H^1(\mathbf{S}^{n-1})\ \text{ for all } \alpha\ge 1\,. \tag{1.7}$$

Regarding the relationship between $L(\log L)^\alpha(\mathbf{S}^{n-1})$ and $H^1(\mathbf{S}^{n-1})$ for $0<\alpha<1$, it is known that neither one is contained in the other. Here, $H^1(\mathbf{S}^{n-1})$ is the Hardy space on the unit sphere in the sense of Coifman and Weiss [5]. Also, recently the authors in [22] proved that

$$B_q^{(0,\upsilon)}(\mathbf{S}^{n-1})\subset L(\log^+ L)^{\upsilon+1}(\mathbf{S}^{n-1})+H^1(\mathbf{S}^{n-1}) \text{ for any } q>1 \text{ and } \upsilon>-1 \tag{1.8}$$

and $B_q^{(0,0)}(\mathbf{S}^{n-1})$ is a proper subspace of $H^1(\mathbf{S}^{n-1})$. The question with regard to the relationship between $B_q^{(0,\upsilon-1)}(\mathbf{S}^{n-1})$ and $L(\log^+ L)^\upsilon(\mathbf{S}^{n-1})$ (for $\upsilon>0$) remains open.

Now let us start recalling some known results relevant to our current work. We start with the following result obtained in [3]:

Theorem A

If $\Omega\in L(\log L)^{1/2}(\mathbf{S}^{n-1})$ then $\mathcal{M}_\Omega$ is bounded on $L^p(\mathbf{R}^n)$ for $1<p<\infty$. Moreover, the exponent $1/2$ is the best possible.

The conclusion of Theorem A for $p=2$ was first obtained by T. Walsh in [20]. Also, T. Walsh proved that the exponent $1/2$ in $L(\log L)^{1/2}(\mathbf{S}^{n-1})$ cannot be replaced by any smaller number. The study of parametric Marcinkiewicz operator $\mathcal{M}^\rho_{\Omega,h}$ was initiated by Hörmander in [11] in which he proved that if $h(r)\equiv 1$, $\rho>0$, and $\Omega\in \mathrm{Lip}_\alpha(\mathbf{S}^{n-1})$ with $\alpha>0$, then $\mathcal{M}^\rho_{\Omega,h}$ is bounded on $L^p(\mathbf{R}^n)$ for $p\in(1,\infty)$. Sakamoto and Yabuta [14] proved $\mathcal{M}^\rho_{\Omega,h}$ is bounded on L^p for $p\in(1,\infty)$ if $h(r)\equiv 1$, $\Omega\in \mathrm{Lip}_\upsilon(\mathbf{S}^{n-1})$ with $0<\upsilon\le 1, h(r)\equiv 1$ and ρ is complex with $\mathrm{Re}(\rho)=\alpha>0$. An improvement of the result in [14] in the case $p=2$ was obtained in [8] as described in the following theorem:

Theorem B

If $h\in\Delta_\gamma(\mathbf{R}_+)$ for some $\gamma>1$ and $\Omega\in L(\log L)(\mathbf{S}^{n-1})$, then $\mathcal{M}^\rho_{\Omega,h}$ is bounded on $L^2(\mathbf{R}^n)$.

On the other hand, when Ω belongs to the block space $B_q^{(0,\upsilon)}(\mathbf{S}^{n-1})$ for some υ and q, the following result was established in [1]:

Theorem C

(a) *If $\Omega\in B_q^{(0,-1/2)}(\mathbf{S}^{n-1})$, $q>1$, then $\mathcal{M}_\Omega$ is bounded on $L^p(\mathbf{R}^n)$ for $1<p<\infty$. Moreover, there exists an Ω which lies in $B_q^{(0,\upsilon)}(\mathbf{S}^{n-1})$ for all $-1<\upsilon<-\frac{1}{2}$ and satisfies (1.1) such that $\mathcal{M}_\Omega$ is not bounded on $L^2(\mathbf{R}^n)$.*

In [7], the authors studied Marcinkiewicz operators $\mathcal{M}^{\rho}_{\mathcal{P},\Omega,h}$ related to polynomial mappings and obtained the following:

Theorem D

Suppose that $h \in L^{\infty}(\mathbf{R}_{+})$, $\rho = 1, \Omega$ *satisfies (1.1) and* $\Omega \in H^{1}\left(\mathbf{S}^{n-1}\right)$. *Then for* $1 < p < \infty$, *there exists a constant* $C_p > 0$ *such that*

$$\left\|\mathcal{M}^{\rho}_{\mathcal{P},\Omega,h}(f)\right\|_{L^{p}(\mathbf{R}^{m})} \leq C_p \|\Omega\|_{H^{1}(\mathbf{S}^{n-1})} \|f\|_{L^{p}(\mathbf{R}^{m})}$$

for any $f \in L^{p}\left(\mathbf{R}^{m}\right)$. *The constant* C_p *may depend on* $n, m, h(\cdot)$ *and* $\deg\left(P_j\right)$, *but it is independent of the coefficients of* $\{P_j\}$.

Our main concern in this paper will be in dealing with Marcinkiewicz operators related to polynomial mappings and under very weak conditions on Ω and h. In fact, we shall prove certain estimates for $\mathcal{M}^{\rho}_{\mathcal{P},\Omega,h}$ and then we apply an extrapolation argument to obtain some new and improved results on Marcinkiewicz integrals. Our conditions on Ω are in the form $\Omega \in L(\log^{+} L)^{\alpha}(\mathbf{S}^{n-1})$ (for some $\alpha > 0$) as well as Ω belongs to certain block spaces $B^{(0,\upsilon)}_{q}\left(\mathbf{S}^{n-1}\right)$ (for $-1 < \upsilon$ and $q > 1$). Historically, in addition to the condition $\Omega \in H^{1}(\mathbf{S}^{n-1})$, these conditions have received a considerable amount of attention with respect to the study of the L^p mapping properties of singular integral operators and Marcinkiewicz integral operators. Our approach in this paper provides an alternative way in dealing with such kind of operators. We should point out that our work is very much motivated by the recent work of S. Sato [15] which in turn was motivated by a remark in [4]. Our main results will be stated in the next section.

2. Main results

In this paper, we establish the following results:

Theorem 2.1

Suppose that $\Omega \in L^{q}(\mathbf{S}^{n-1})$ *for some* $q \in (1,2]$ *and* $h \in \Delta_{\gamma}(\mathbf{R}_{+})$ *for some* $1 < \gamma \leq \infty$. *Then*

$$\left\|\mathcal{M}^{\rho}_{\mathcal{P},\Omega,h} f\right\|_{L^{p}(\mathbf{R}^{m})} \leq C_p (q-1)^{-1/2} A(\gamma) \|\Omega\|_{L^{q}(\mathbf{S}^{n-1})} \|h\|_{\Delta_{\gamma}} \|f\|_{L^{p}(\mathbf{R}^{m})} \qquad (2.1)$$

for $|1/p - 1/2| < \min\{1/2, 1/\gamma'\}$, *where* $A(\gamma) = \begin{cases} \gamma^{1/2} & \text{if } \gamma > 2, \\ (\gamma-1)^{-1/2} & \text{if } 1 < \gamma \leq 2 \end{cases}$ *and* C_p *is a constant which may depend on* n, m, γ *and* $\deg\left(P_j\right)$, *but is independent of the coefficients of* $\{P_j\}$, Ω, γ *and* q.

Remark 2 We notice that if $\gamma \geq 2$, the range of p in Theorem 2.1 is the full range $(1, \infty)$, whereas this range becomes progressively smaller as $\gamma \to 1^{+}$. In our next theorem one will see that the L^p boundedness holds for the full range $1 < p < \infty$ when $\mathcal{P}$ lies in a special class of polynomial mappings (denoted by $\mathcal{F}(n,m)$), irrespective of how close γ is to 1. This is true, in particular, when every P_j is odd.

Theorem 2.2

Suppose that $\mathcal{P} \in F(n,m)$, Ω *satisfies (1.1),* $\Omega \in L^q(\mathbf{S}^{n-1})$ *for some* $q \in (1,2]$ *and* $h \in \Delta_\gamma(\mathbf{R}_+)$ *for some* $\gamma \in (1,2]$. *Then*

$$\left\|\mathcal{M}^{\rho}_{\mathcal{P},\Omega,h}f\right\|_{L^p(\mathbf{R}^m)} \le C_p(q-1)^{-1/2}(\gamma-1)^{-1/2}\|\Omega\|_{L^q(\mathbf{S}^{n-1})}\|h\|_{\Delta_\gamma}\|f\|_{L^p(\mathbf{R}^m)} \text{ if } 2 \le p < \infty \tag{2.2}$$

and

$$\left\|\mathcal{M}^{\rho}_{\mathcal{P},\Omega,h}f\right\|_{L^p(\mathbf{R}^m)} \le C_p(q-1)^{-1}(\gamma-1)^{-1}\|\Omega\|_{L^q(\mathbf{S}^{n-1})}\|h\|_{\Delta_\gamma}\|f\|_{L^p(\mathbf{R}^m)} \text{ if } 1 < p < 2, \tag{2.3}$$

where the constant C_p *is a positive constant independent of* γ, q, Ω *and* h. *Furthermore, if* $\mathcal{P}(-x) = -\mathcal{P}(x)$, $x \in \mathbf{R}^n$, *then the constant* C_p *may depend on* n, m, $\deg(P_j)$, *but is independent of the coefficients of* $\{P_j\}$.

By the conclusion in Theorems 2.1–2.2 and extrapolation, we respectively get the following results:

Theorem 2.3

Suppose that Ω *satisfies (1.1) and* $h \in \Delta_\gamma(\mathbf{R}_+)$ *for some* $\gamma > 1$.

(a) *If* $\Omega \in L(\log L)^{1/2}(\mathbf{S}^{n-1})$, *then*

$$\left\|\mathcal{M}^{\rho}_{\mathcal{P},\Omega,h}f\right\|_{L^p(\mathbf{R}^m)} \le C_pA(\gamma)\left(1+\|\Omega\|_{L(\log L)^{1/2}(\mathbf{S}^{n-1})}\right)\|h\|_{\Delta_\gamma}\|f\|_{L^p(\mathbf{R}^m)}$$

for $|1/p-1/2| < \min\{1/2, 1/\gamma'\}$;

(b) *If* $\Omega \in B_q^{(0,-1/2)}(\mathbf{S}^{n-1})$ *for some* $q > 1$, *then*

$$\left\|\mathcal{M}^{\rho}_{\mathcal{P},\Omega,h}f\right\|_{L^p(\mathbf{R}^m)} \le C_pA(\gamma)\left(1+\|\Omega\|_{B_q^{(0,-1/2)}(\mathbf{S}^{n-1})}\right)\|h\|_{\Delta_\gamma}\|f\|_{L^p(\mathbf{R}^m)}$$

for $|1/p-1/2| < \min\{1/2, 1/\gamma'\}$. *The constant* C_p *is independent of* Ω, h *and the coefficients of* $\{P_j\}$.

Theorem 2.4

Suppose that $\mathcal{P} \in F(n,m)$ *and* Ω *satisfies (1.1).*

(a) *If* $\Omega \in L(\log L)^{1/2}(\mathbf{S}^{n-1})$ *and* $h \in \mathcal{N}_{1/2}(\mathbf{R}_+)$, *then*

$$\left\|\mathcal{M}^{\rho}_{\mathcal{P},\Omega,h}f\right\|_{L^p(\mathbf{R}^m)} \le C_p\left(1+\|\Omega\|_{L(\log L)^{1/2}(\mathbf{S}^{n-1})}\right)(1+N_{1/2}(h))\|f\|_{L^p(\mathbf{R}^m)}$$

for $2 \le p < \infty$;

(b) *If* $\Omega \in L(\log L)(\mathbf{S}^{n-1})$ *and* $h \in \mathcal{N}_1(\mathbf{R}_+)$, *then*

$$\left\|\mathcal{M}^{\rho}_{\mathcal{P},\Omega,h}f\right\|_{L^p(\mathbf{R}^m)} \le C_p\left(1+\|\Omega\|_{L(\log L)(\mathbf{S}^{n-1})}\right)(1+N_1(h))\|f\|_{L^p(\mathbf{R}^m)}$$

for $1 < p < 2$, *where the constant* C_p *is independent of* Ω, h *and the coefficients of* $\{P_j\}$. *Furthermore, if* $\mathcal{P}(-x) = -\mathcal{P}(x)$, $x \in \mathbf{R}^n$, *then the constant* C_p *may depend on* n, m $\deg(P_j)$, *but it is independent of the coefficients of* $\{P_j\}$.

Theorem 2.5

Suppose that $\mathcal{P} \in F(n,m)$ and Ω satisfies (1.1).

(a) *If $\Omega \in B_q^{(0,-1/2)}(\mathbf{S}^{n-1})$ for some $q > 1$ and $h \in \mathcal{N}_{1/2}(\mathbf{R}_+)$, then*

$$\left\|\mathcal{M}^{\rho}_{\mathcal{P},\Omega,h} f\right\|_{L^p(\mathbf{R}^m)} \le C_p\Big(1 + \|\Omega\|_{B_q^{(0,-1/2)}(\mathbf{S}^{n-1})}\Big)(1 + N_{1/2}(h))\,\|f\|_{L^p(\mathbf{R}^m)}$$

for $2 \le p < \infty$;

(b) *If $\Omega \in B_q^{(0,0)}(\mathbf{S}^{n-1})$ for some $q > 1$ and $h \in \mathcal{N}_1(\mathbf{R}_+)$, then*

$$\left\|\mathcal{M}^{\rho}_{\mathcal{P},\Omega,h} f\right\|_{L^p(\mathbf{R}^m)} \le C_p\Big(1 + \|\Omega\|_{B_q^{(0,0)}(\mathbf{S}^{n-1})}\Big)(1 + N_1(h))\,\|f\|_{L^p(\mathbf{R}^m)}$$

for $1 < p < 2$, where the constant C_p is independent of Ω, h and the coefficients of $\{P_j\}$. Furthermore, if $\mathcal{P}(-x) = -\mathcal{P}(x)$, $x \in \mathbf{R}^n$, then the constant C_p may depend on n, m, $\deg(P_j)$, but it is independent of the coefficients of $\{P_j\}$.

Remark 3

(1) By the conclusion in Theorem 2.4 (respectively Theorem 2.5) along with Remark 1 (5), we conclude that $\mathcal{M}^{\rho}_{\mathcal{P},\Omega,h} f$ is bounded on L^p if p and Ω are given as in Theorem 2.4 (respectively Theorem 2.5) and if $h \in \mathcal{L}_\gamma(\mathbf{R}_+)$ for some $\gamma > \frac{3}{2}$ in Part (a) and $h \in \mathcal{L}_\gamma(\mathbf{R}_+)$ for some $\gamma > 2$ in Part (b) in Theorem 2.4 (respectively Theorem 2.5).

(2) Theorems 2.3-2.5 greatly generalize the main results in [1, 3].

(3) It is known that the conditions $\Omega \in B_q^{(0,-1/2)}(\mathbf{S}^{n-1})$ $(q > 1)$ and $\Omega \in L(\log L)^{1/2}(\mathbf{S}^{n-1})$ are optimal for the L^2 boundedness of $\mathcal{M}_\Omega$ to hold in the sense that $1/2$ cannot be replaced by any smaller number. Also, the conditions imposed on h in Theorems 2.4-2.5 are the weakest known conditions.

(4) It is clear in Theorems 2.4 and 2.5 the conditions on Ω and h in part (a) are considerably better than the conditions on Ω and h in part (b). It would be an interesting problem to determine whether these conditions on Ω and h in part (b) can be weakened to be as in part (a) in both Theorems 2.4 and 2.5.

(5) The method employed in this paper is based in part on ideas from [9, 10, 4, 2, 15, 1].

Marcinkiewicz integral operators belong to the broad class of Littlewood-Paley g-functions and L^p bounds regarding them are useful in the study of smoothness properties of functions and behavior of integral transformations, such as Poisson integrals, singular integrals and, more generally, singular Radon transforms. While interest in them dates back several decades, recent efforts have been mostly focused on finding the weakest possible kernel conditions under which L^p boundedness holds. The paper [3] took a big step in this direction and the current paper represents a significant improvement and expansion of it.

Throughout the rest of the paper the letter C denotes a positive whose value may be different at each appearance.

The authors wish to thank the referee for his helpful comments.

3. Definitions and lemmas

We start this section by introducing some notation. For a positive integer m, we let $L(\mathbf{R}^n, \mathbf{R}^m)$ denote the space of linear transformations from $\mathbf{R}^n$ into $\mathbf{R}^m$, $\mathcal{V}_m$ denote the linear space of real-valued homogeneous polynomials of degree m on $\mathbf{R}^n$ with $\alpha_m = \dim(\mathcal{V}_m)$ and $\mathcal{A}_n$ be the class of polynomials of n variables with real coefficients. For $\mathcal{P} = (P_1, \dots, P_m) \in (\mathcal{A}_n)^m$, we shall use $\deg(\mathcal{P})$ to denote $\max_{1\le k\le m} \deg(P_k)$ and for $P(y) = \sum_{|\alpha|=m} a_\alpha y^\alpha \in \mathcal{V}_m$, we set $\|P\| = \sum_{|\alpha|=m} |a_\alpha|$. If m is an even, positive integer, then we have $|x|^m = (x_1^2 + x_2^2 + \cdots + x_n^2)^{m/2} \in \mathcal{V}_m$. We now choose a basis $\{\eta_1, \dots, \eta_{\alpha_m}\}$ for the space $\mathcal{V}_m$ such that $\eta_1(x) = |x|^m$ for $x \in \mathbf{R}^n$. It is clear that there are constants C_1 and C_2 such that $C_1\big(\sum_{j=1}^{\alpha_m} |c_j|\big) \le \|P\| \le C_2\big(\sum_{j=1}^{\alpha_m} |c_j|\big)$ for every $P = \sum_{j=1}^{\alpha_m} c_j\eta_j \in \mathcal{V}_m$.We define the linear transformation $Y_m : \mathcal{V}_m \to \mathcal{V}_m$ by $Y_m(P) = \sum_{j=2}^{\alpha_m} c_j\eta_j$ for $P = \sum_{j=1}^{\alpha_m} c_j\eta_j$. Also, define the linear transformation $Z_m^n : \mathcal{V}_m \to \mathcal{V}_m$ by

$$Z_m^n = \begin{cases} id_{\alpha_m} & \text{if } m \text{ is odd} \\ Y_m & \text{if } m \text{ is even.} \end{cases}$$

Now we give the definition of the special class of polynomial mappings $\mathcal{F}(n,m)$. This class of functions was introduced by Fan and Pan in [10]. It is defined as follows: for $n, m, l \in \mathbf{N}$, let $\mathcal{F}_{n,m,0} = \mathbf{R}^m$,

$$\mathcal{F}_{n,m,l} = \Big\{(P_1, \dots, P_m) \in (\mathcal{V}_l)^m : |x|^l \notin \operatorname{span}\{P_1, \dots, P_m\}\Big\},$$

$$\mathcal{F}(n,m) = \left\{\sum_{l=0}^{m} P_j^l : m \ge 0, \mathcal{P}^l \in \mathcal{F}_{n,m,l} \text{ for } 0 \le l \le m\right\}.$$

It is clear that $\mathcal{F}_{n,m,l} = (\mathcal{V}_l)^m$ if l is odd. Also, notice that if $\mathcal{P} = (P_1, \dots, P_m)$ with $P_j \in \mathcal{A}_n$ and $\mathcal{P}(-x) = -\mathcal{P}(x)$, then $\mathcal{P} \in \mathcal{F}(n,m)$.

For $1 \le m \le n$, we define the projection operator $\pi_m^n : \mathbf{R}^n \to \mathbf{R}^m$ by $\pi_m^n(\xi) = (\xi_1, \dots, \xi_m)$. Also, let $t^{\pm\alpha} = \inf(t^\alpha, t^{-\alpha})$.

DEFINITION 3.1 Let $\theta \ge 2$. For a suitable mapping $\Phi : \mathbf{R}^n \to \mathbf{R}^m$, a measurable function $h : \mathbf{R}_+ \longrightarrow \mathbf{C}$ and $\Omega : \mathbf{S}^{n-1} \to \mathbf{R}$, we define the family of measures $\{\sigma_{t,\Phi,h} : t \in \mathbf{R}_+\}$ and the related maximal operators $\sigma^*_{\Phi,h}$ and $M_{h,\theta,\Phi}$ on $\mathbf{R}^m$ by

$$\int_{\mathbf{R}^m} f d\sigma_{t,\Phi,h} = \frac{1}{t^\rho}\int_{1/2t<|u|\le t} f(\Phi(u))h(|u|)\frac{\Omega(u')}{|u|^{n-\rho}}du,$$

$$\sigma^*_{\Phi,h}f(x) = \sup_{t\in\mathbf{R}_+} ||\sigma_{t,\Phi,h}| * f(x)|,$$

$$M_{h,\theta,\Phi}f(x) = \sup_{k\in\mathbf{Z}} \int_{\theta^k}^{\theta^{k+1}} ||\sigma_{t,\Phi,h}| * f(x)|\, dt/t,$$

where $|\sigma_{t,\Phi,h}|$ is defined in the same way as $\sigma_{t,\Phi,h}$, but with Ω replaced by $|\Omega|$ and h replaced by $|h|$.

The following result follows from Lemmas 3.3-3.4, 3.7 and Remark 3.6 in [10].

Lemma 3.2

Let $m \in \mathbf{N}$. Then there exists a positive constant $A_{m,\varepsilon}$ such that

$$\sup_{\lambda\in\mathbf{R}} \int_{\mathbf{S}^{n-1}} |P(y)-\lambda|^{-\varepsilon}\, d\sigma(y) \le A_{m,\varepsilon} \, \|Z_m^n(P)\|^{-\varepsilon} \tag{3.1}$$

for every $P \in \mathcal{V}_m$, and $\varepsilon \in [0, \varepsilon(m))$, where $\varepsilon(d) = \frac{2}{[3+(-1)^{d+1}]d}$. If U is a subspace of $\mathcal{V}_m$ satisfying $|x|^m \notin U$, then there exists a constant $A'_{m,\varepsilon}$ such that

$$\sup_{\lambda\in\mathbf{R}} \int_{\mathbf{S}^{n-1}} |P(y)-\lambda|^{-\varepsilon}\, d\sigma(y) \le A'_{d,\varepsilon} \, \|P\|^{-\varepsilon} \tag{3.2}$$

holds for $\varepsilon \in [0, \varepsilon(m))$ and all $P \in U$. The constant $A'_{m,\varepsilon}$ may depend on the subspace U if m is even, but it is independent of U if m is odd.

We shall need the following lemma which has its roots in [9, 10, 2]. A proof of this lemma can be obtained by the same proof (with only minor modifications) as that of Lemma 3.2 in [2]. We omit the details.

Lemma 3.3

Let $\{\sigma_k : k \in \mathbf{Z}\}$ be a sequence of Borel measures on $\mathbf{R}^n$. Let $L : \mathbf{R}^n \to \mathbf{R}^m$ be a linear transformation. Suppose that for all $k \in \mathbf{Z}$, $\xi \in \mathbf{R}^n$, for some $a \ge 2$, $\alpha, C > 0$, $A > 1$ and $p_0 \in (2, \infty)$ we have:

(i) $|\hat{\sigma}_k(\xi)| \le CA(a^k \, |L(\xi)| \,)^{\pm\alpha/(\log(a))}$;

(ii)

$$\Big\| \Big(\sum_{k\in\mathbf{Z}} |\sigma_k * g_k|^2 \Big)^{1/2} \Big\|_{p_0} \le CA \Big\| \Big(\sum_{k\in\mathbf{Z}} |g_k|^2 \Big)^{1/2} \Big\|_{p_0}$$

for arbitrary functions $\{g_k\}$ on $\mathbf{R}^n$. Then for $p_0' < p < p_0$, there exists a constant $C_p > 0$ such that the inequality

$$\Big\| \Big(\sum_{k\in\mathbf{Z}} |\sigma_k * f|^2 \Big)^{1/2} \Big\|_{p} \le C_p A \, \|f\|_p$$

holds for all f in $L^p(\mathbf{R}^n)$. The constant C_p is independent of A and the linear transformation L.

The proof of Theorems 2.1 and 2.2 will rely heavily on the following lemma:

Lemma 3.4

Let $a \ge 2$, $A > 1$ and $C > 0$. Let $\{\sigma_t : t \in \mathbf{R}_+\}$ be a family of Borel measures on $\mathbf{R}^n$ and let L: $\mathbf{R}^n \to \mathbf{R}^m$ be a linear transformation. Suppose that we have the following:

(i) $\Big(\int_{a^k}^{a^{k+1}} |\hat{\sigma}_t(\xi)|^2 \, dt/t\Big)^{1/2} \le CA(a^k \, |L(\xi)|)^{\pm\alpha/(\log(a))}$;

(ii) *for some* $p_0 \in (1, \infty)$,

$$\Big\| \Big(\sum_{k\in\mathbf{Z}} \int_{a^k}^{a^{k+1}} |\sigma_t * g_k|^2 \, dt/t \Big)^{1/2} \Big\|_{p_0} \le C_{p_0} A \Big\| \Big(\sum_{k\in\mathbf{Z}} |g_k|^2 \Big)^{1/2} \Big\|_{p_0}$$

holds for all functions $\{g_k\}$ on $\mathbf{R}^n$.

Then for every p with either $p \in [2, p_0)$ if $p_0 \in [2, \infty)$ or $p \in (p_0, 2]$ if $p_0 \in (1, 2]$, there exists a positive constant C_p such that

$$\left\| \Big(\int_0^\infty |\sigma_t * f|^2 \, dt/t \Big)^{1/2} \right\|_p \leq C_p A \left\| f \right\|_p$$

for all $f \in L^p(\mathbf{R}^n)$. The constant C_p is independent of A. Here we use the convention $p \in [2, 2) \cup (2, 2]$ to mean that $p = 2$.

Proof. Without loss of generality, we may assume that $0 < \alpha \leq 1$. By the arguments in the proof of Lemma 6.2 in [10], we may assume without loss of generality that $m \leq n$ and $L = \pi_m^n$. Let $Hf(x) = \Big(\int_0^\infty |\sigma_t * f(x)|^2 \, dt/t \Big)^{1/2}$ and let $\{\psi_{k,a}\}_{-\infty}^{\infty}$ be a sequence in $C^\infty((0, \infty))$ such that

$$0 \leq \psi_{k,a} \leq 1, \ \sum_k \psi_{k,a}(t) = 1, \ \text{supp } \psi_{k,a} \subseteq [a^{-k-1}, a^{-k+1}], \ |(d/dt)^j \psi_{k,a}(t)| \leq \frac{C_j}{t^j},$$

where $t > 0, k \in \mathbf{Z}$, $j \in \mathbf{N}$ and the constants C_k are independent of a. For $k \in \mathbf{Z}$, let $T_{k,a}$ be the operator defined by $(\widehat{T_{k,a} f})(\xi) = \psi_{k,a}(|\pi_m^n \xi|) \hat{f}(\xi)$ for $\xi \in \mathbf{R}^n$. Then for any $f \in \mathcal{S}(\mathbf{R}^n)$ and $j \in \mathbf{Z}$ we have $f(x) = \sum_{k \in \mathbf{Z}} (T_{k+j,a} f)(x)$. Thus by using Minkowski's inequality we get

$$\mathcal{H}(f) \leq \sum_{j \in \mathbf{Z}} \mathcal{H}_j(f), \tag{3.3}$$

where

$$\mathcal{H}_j f(x) = \Big(\sum_{k \in \mathbf{Z}} \int_{a^k}^{a^{k+1}} |(\sigma_t * T_{k+j,a} f)(x)|^2 \, dt/t \Big)^{1/2} \text{ and } f \in \mathcal{S}(\mathbf{R}^n).$$

So it is clear that we only need to estimate $\left\| \mathcal{H}_j(f) \right\|_p$ for any j. First, we compute $\left\| \mathcal{H}_j(f) \right\|_p$ for $p = p_0$.

$$\begin{aligned} \left\| \mathcal{H}_j f \right\|_{p_0} &= \left\| \Big(\sum_{k \in \mathbf{Z}} \int_{a^k}^{a^{k+1}} |(\sigma_t * T_{k+j,a} f)|^2 \, dt/t \Big)^{1/2} \right\|_{p_0} \\ &\leq C_{p_0} A \left\| \Big(\sum_{k \in \mathbf{Z}} |T_{k+j,a} f|^2 \Big)^{1/2} \right\|_{L^{p_0}} \\ &\leq CA \left\| f \right\|_{p_0}, \end{aligned} \tag{3.4}$$

where the first inequality follows by using condition (ii) and the second inequality follows by using Littlewood-Paley theory and Theorem 3 along with the remark that follows its statement in [17, p. 96].

Now, we compute $\left\| \mathcal{H}_j(f) \right\|_2$. Let $\Delta_j = \{\xi : a^{-(j+1)} \leq |\pi_m^n \xi| \leq a^{-(j-1)}\}$. By

Plancherel's theorem

$$\begin{aligned}
\|\mathcal{H}_j(f)\|_2^2 &= \sum_{k\in\mathbf{Z}}\int_{\mathbf{R}^n} |\psi_{k,a}(|\pi_m^n\xi|)|^2 \left|\hat{f}(\xi)\right|^2 \Big(\int_{a^k}^{a^{k+1}} |\hat{\sigma}_t(\xi)|^2\,dt/t\Big)d\xi \\
&= \sum_{k\in\mathbf{Z}}\int_{\Delta_{j+k}} \Big(\int_{a^k}^{a^{k+1}} |\hat{\sigma}_t(\xi)|^2\,dt/t\Big)\left|\hat{f}(\xi)\right|^2 d\xi \\
&\le CA^2\sum_{k\in\mathbf{Z}}\int_{\Delta_{j+k}} \left|a^k\pi_m^n\xi\right|^{\pm\alpha/(\log a)} \left|\hat{f}(\xi)\right|^2 d\xi \\
&\le CA^2 e^{-\alpha(|j|-1)}\sum_{k\in\mathbf{Z}}\int_{\Delta_{j+k}} \left|\hat{f}(\xi)\right|^2 d\xi \\
&\le CA^2 e^{\alpha}e^{-\alpha|j|}\,\|f\|_2^2
\end{aligned}$$

and hence

$$\|\mathcal{H}_j(f)\|_2 \le CAe^{-\alpha/2|j|}\,\|f\|_2\,. \tag{3.5}$$

By interpolating between (3.4) and (3.5) we get

$$\|\mathcal{H}_j(f)\|_p \le C(\alpha,p)Ae^{-\alpha_p|j|}\,\|f\|_p\,, \tag{3.6}$$

where $C(\alpha,p)$ and α_p are positive constants independent of a, A and j. Lemma 3.4 now follows from (3.3) and (3.6).

Lemma 3.5

Let $N\in\mathbf{N}$ and $\{\sigma_t^{(l)} : t\in\mathbf{R}_+,\ 0\le l\le N\}$ be a family of Borel measures on $\mathbf{R}^n$ with $\sigma_t^{(0)}=0$ for all $t\in\mathbf{R}_+$. Let $\{a_l : 1\le l\le N\}\subseteq \mathbf{R}_+\backslash(0,2)$, $\{m_l : 1\le l\le N\}\subseteq\mathbf{N}$, $\{\alpha_l :\ 1\le l\le N\}\subseteq\mathbf{R}_+$, and let $L_l\in L(\mathbf{R}^n,\mathbf{R}^{m_l})$ for $1\le l\le N$. Suppose that for all $t\in\mathbf{R}_+$, $1\le l\le N$, for all $\xi\in\mathbf{R}^n$ and for some constants $C>0$ and $A>0$ we have the following:

(i) $\Big(\int_{a_l^k}^{a_l^{k+1}} \left|\hat{\sigma}_t^{(l)}(\xi)\right|^2 dt/t\Big)^{1/2} \le CA\left|a_l^k L_l(\xi)\right|^{-\alpha_l/(\log(a_l))}$;

(ii) $\Big(\int_{a_l^k}^{a_l^{k+1}} \left|\hat{\sigma}_t^{(l)}(\xi)-\hat{\sigma}_t^{(l-1)}(\xi)\right|^2 dt/t\Big)^{1/2} \le CA\left|a_l^k L_l(\xi)\right|^{\alpha_l/(\log(a_l))}$;

(iii) *for some $p_0\in(1,\infty)$,*

$$\left\|\Big(\sum_{k\in\mathbf{Z}}\int_{a_l^k}^{a_l^{k+1}} \left|\sigma_t^{(l)} * g_k\right|^2 dt/t\Big)^{1/2}\right\|_{p_0} \le C_{p_0}A\left\|\Big(\sum_{k\in\mathbf{Z}}|g_k|^2\Big)^{1/2}\right\|_{p_0}$$

holds for arbitrary functions $\{g_k\}$ on $\mathbf{R}^n$.

Then for every p with either $p\in[2,p_0)$ if $p_0\in[2,\infty)$ or $p\in(p_0,2]$ if $p_0\in(1,2]$, there exists a positive constant C_p such that

$$\left\|\Big(\int_0^\infty \left|\sigma_t^{(N)} * f\right|^2 dt/t\Big)^{1/2}\right\|_p \le C_pA\,\|f\|_p \tag{3.7}$$

for all $f\in L^p(\mathbf{R}^n)$. The constant C_p is independent of A and the linear transformations $\{L_l\}_{l=1}^N$.

Proof. The idea of the proof is similar to the one appearing in the proof of Theorem 7.6 in [10]. Without loss of generality, we may assume that $0 < \alpha_l \leq 1$, $m_l \leq n$ and $L_l(\xi) = (\xi_1, ..., \xi_{m_l})$ for $\xi = (\xi_1, ..., \xi_n) \in \mathbf{R}^n$ and $1 \leq l \leq N$. Define the family of measures $\{\mu_t^{(l)} : 1 \leq l \leq N, t \in \mathbf{R}_+\}$ as follows: choose and fix a function $\varphi \in C_0^\infty(\mathbf{R})$ such that $\varphi(s) = 1$ for $|t| \leq \frac{1}{2}$ and $\varphi(s) = 0$ for $|t| \geq 1$. Let $\psi(t) = \varphi(t^2)$ and for $t \in \mathbf{R}_+$, let

$$\hat{\mu}_t^{(l)}(\xi) = \hat{\sigma}_t^{(l)}(\xi) \prod_{l<j\leq N} \psi(a_j^k |L_l(\xi)|) - \hat{\sigma}_t^{(l-1)}(\xi) \prod_{l-1<j\leq N} \psi(a_j^k |L_l(\xi)|) \tag{3.8}$$

when $1 \leq l \leq N-1$ and

$$\hat{\mu}_t^{(N)}(\xi) = \hat{\sigma}_t^{(N)}(\xi) - \hat{\sigma}_t^{(N-1)}(\xi)\psi(a_N^k |L_l(\xi)|). \tag{3.9}$$

By straightforward calculations, conditions (i)-(ii) and (3.8)–(3.9) we get

$$\Big(\int_{a_l^k}^{a_l^{k+1}} \left| \hat{\mu}_t^{(l)}(\xi) \right|^2 dt/t \Big)^{1/2} \leq \; CA(a_l^k |L_l(\xi)|)^{\pm\alpha_l/(\log(a_l))} \text{ for all } 1 \leq l \leq N. \tag{3.10}$$

By condition (iii), it is easy to see that

$$\left\| \Big(\sum_{k\in\mathbf{Z}} \int_{a_l^k}^{a_l^{k+1}} \left| \sigma_t^{(l)} * g_k \right|^2 dt/t \Big)^{1/2} \right\|_{L^{p_0}(\mathbf{R}^n)} \leq C_{p_0} A \left\| \Big(\sum_{k\in\mathbf{Z}} |g_k|^2 \Big)^{1/2} \right\|_{L^{p_0}(\mathbf{R}^m)} \tag{3.11}$$

holds for all functions $\{g_k\}$ on $\mathbf{R}^n$, for $1 < p < \infty$, $f \in L^p(\mathbf{R}^n)$ and $1 \leq l \leq N$. By (3.10)-(3.11) and invoking Lemma 3.4, for $1 \leq l \leq N$ and for each p given as in the statement of the lemma, there exists a positive constant C_p such that

$$\left\| \Big(\int_0^\infty \left| \mu_t^{(l)} * f \right|^2 dt/t \Big)^{1/2} \right\|_p \leq C_p A \|f\|_p \tag{3.12}$$

holds for all f in $L^p(\mathbf{R}^n)$. Since $\sigma_t^{(0)} = 0$, we find that $\sigma_k^{(N)} = \sum_{l=1}^N \mu_k^{(l)}$ and hence by (3.12) we get (3.7). The proof of Lemma 3.5 is complete.

Lemma 3.6

Let $\theta \geq 2$, $h \in \Delta_\gamma(\mathbf{R}_+)$ for some $\gamma > 1$ and $\Omega \in L^q(\mathbf{S}^{n-1})$ for some $q > 1$. Suppose $F : \mathbf{R}^n \to \mathbf{R}$ is a function given by $F(x) = \sum_{j=0}^l P_j(x) + W(|x|)$, where $P_j(\cdot)$ is a homogeneous polynomial of degree j for $0 \leq j \leq l$ and $W(\cdot)$ is an arbitrary function. Let

$$I_t(\Omega, F) = \int_{1/2t}^t \int_{\mathbf{S}^{n-1}} e^{iF(sx)} \Omega(x) h(s) d\sigma(x)\, ds/s \quad \text{with} \quad t \in \mathbf{R}_+ .$$

Then there exist positive constants C and α independent of θ such that

$$\int_{\theta^k}^{\theta^{k+1}} |I_t(\Omega, F)|^2 dt/t \leq C(\log\theta) \|\Omega\|_q^2 \|h\|_{\Delta_\gamma}^2 ; \tag{3.13}$$

$$\int_{\theta^k}^{\theta^{k+1}} |I_t(\Omega, F)|^2 dt/t \leq C(\log\theta) \|\Omega\|_q^2 \|h\|_{\Delta_\gamma}^2 \left(\theta^{lk} \|Z_l(P_l)\| \right)^{\alpha/(l\gamma' q')} ; \tag{3.14}$$

holds for all $k \in \mathbf{Z}$. The constant C is independent of k, Ω, $W(\cdot)$ and the coefficients of $P_j(\cdot)$. If G is a subspace of V_l satisfying $|x|^l \notin G$ for some $l \in \mathbf{N}$, then there exists a constant C' such that

$$\int_{\theta^k}^{\theta^{k+1}} |I_t(\Omega, F)|^2 \, dt/t \le C(\log\theta) \, \|\Omega\|_q^2 \, \|h\|_{\Delta_\gamma}^2 \, ; \tag{3.15}$$

$$\int_{\theta^k}^{\theta^{k+1}} |I_t(\Omega, F)|^2 \, dt/t \le C(\log\theta) \, \|\Omega\|_q^2 \, \|h\|_{\Delta_\gamma}^2 \left(\theta^{lk} \, \|P_l\|\right)^{\alpha/(l\gamma' q')} ; \tag{3.16}$$

holds for all $k \in \mathbf{Z}$ and $P_l \in G$. The constant C' may depend on the subspace G if l is even, but it is independent of G if l is odd.

Proof. It is easy to verify that (3.13) and (3.15) hold. First By a change of variable and Hölder's inequality we have

$$|J_t(\Omega, F)| \le C \, \|h\|_{\Delta_\gamma} \Big(\int_{1/2}^1 \Big| \int_{\mathbf{S}^{n-1}} \Omega(x) e^{-iF(stx)} d\sigma(x) \Big|^{\gamma'} ds/s \Big)^{1/\gamma'} .$$

If $1 < \gamma \le 2$, by noticing that

$$\Big| \int_{\mathbf{S}^{n-1}} \Omega(x) e^{-iF(sx)} d\sigma(x) \Big| \le \|\Omega\|_{L^1(\mathbf{S}^{n-1})} ,$$

we obtain

$$|J_t(\Omega, F)| \le \|h\|_{\Delta_\gamma} \, \|\Omega\|_{L^1(\mathbf{S}^{n-1})}^{(\gamma'-2)/\gamma'} \Big(\int_{1/2}^1 \Big| \int_{\mathbf{S}^{n-1}} \Omega(x) e^{-iF(stx)} d\sigma(x) \Big|^2 ds/s \Big)^{1/\gamma'} .$$

If $\gamma > 2$, by Hölder's inequality, we get

$$|J_t(\Omega, F)| \le \|h\|_{\Delta_\gamma} \Big(\int_{1/2}^1 \Big| \int_{\mathbf{S}^{n-1}} \Omega(x) e^{-iF(stx)} d\sigma(x) \Big|^2 ds/s \Big)^{1/2} .$$

Thus, in either case we have

$$\begin{aligned} |J_t(\Omega, F)| \le\ & \|h\|_{\Delta_\gamma} \, \|\Omega\|_{L^1(\mathbf{S}^{n-1})}^{(\max\{2,\gamma'\}-2)/\gamma'} \\ & \times \Big(\int_{1/2}^1 \Big| \int_{\mathbf{S}^{n-1}} \Omega(x) e^{-iF(stx)} d\sigma(x) \Big|^2 ds/s \Big)^{1/(\max\{2,\gamma'\})} . \end{aligned}$$

Therefore,

$$\begin{aligned} |J_t(\Omega, F)| \le\ & \|h\|_{\Delta_\gamma} \, \|\Omega\|_{L^1(\mathbf{S}^{n-1})}^{(\max\{2,\gamma'\}-2)/\gamma'} \\ & \times \left(\int_{\mathbf{S}^{n-1} \times \mathbf{S}^{n-1}} \Omega(x) \overline{\Omega(y)} \Big(\int_{1/2}^1 e^{i(F(stx)-F(sty))} ds/s \Big) d\sigma(x) \, d\sigma(y) \right)^{1/(\max\{2,\gamma'\})} . \end{aligned} \tag{3.17}$$

By Van der Corput's lemma we obtain

$$\begin{aligned} \Big| \int_{1/2}^1 e^{i(F(stx)-F(sty))} ds/s \Big| & \le C \min\big\{ \log 2, |t^l(P_l(x) - P_l(y))|^{-1/l} \big\} \\ & \le C \begin{cases} |t^l(P_l(x) - P_l(y))|^{-1/(4lq')}, & \text{if } 1 < \gamma \le 2, \\ |t^l(P_l(x) - P_l(y))|^{-1/(4l\gamma' q')}, & \text{if } \gamma > 2. \end{cases} \end{aligned} \tag{3.18}$$

By Hölder's inequality, (3.17)–(3.18) and applying Lemma 3.2 we get

$$\begin{aligned}|J_t(\Omega,F)| &\leq C\,\|h\|_{\Delta_\gamma}\,\|\Omega\|_{L^1(\mathbf{S}^{n-1})}^{(\max\{2,\gamma'\}-2)/\gamma'}\,\|\Omega\|_{L^q(\mathbf{S}^{n-1})}^{2/(\max\{2,\gamma'\})}\left(t^l\,\|Z_l(P_l)\|\,\right)^{-1/(4l\gamma'q')}\\ &\leq C\,\|h\|_{\Delta_\gamma}\,\|\Omega\|_{L^q(\mathbf{S}^{n-1})}\left(t^l\,\|Z_l(P_l)\|\,\right)^{-1/(4l\gamma'q')}\end{aligned}$$

which easily implies (3.14). A proof of (3.16) can be constructed by following essentially the same argument as in the proof of (3.14). Lemma 3.6 is proved.

Lemma 3.7

Let $h \in \Delta_\gamma(\mathbf{R}_+)$ for some $1<\gamma\leq 2$, $\Omega\in L^q(\mathbf{S}^{n-1})$ for some $1<q\leq 2$ and $\theta = 2^{q'\gamma'}$. Let $\mathcal{P}=(P_1,...,P_m)$ be a polynomial mapping from $\mathbf{R}^n$ into $\mathbf{R}^m$. Then for any p satisfying $|1/p-1/2|<1/\gamma'$ and $f\in L^p(\mathbf{R}^m)$, there exists a positive constant C_p which is independent of q,γ,Ω and h such that

$$\begin{aligned}&\left\|\Big(\sum_{k\in\mathbf{Z}}\int_{\theta^k}^{\theta^{k+1}}|\sigma_{t,\mathcal{P},h}*g_k|^2\,dt/t\Big)^{1/2}\right\|_{L^p(\mathbf{R}^m)}\\ &\qquad\leq C_p(q-1)^{-1/2}(\gamma-1)^{-1/2}\,\|\Omega\|_{L^q(\mathbf{S}^{n-1})}\,\|h\|_{\Delta_\gamma}\left\|\Big(\sum_{k\in\mathbf{Z}}|g_k|^2\Big)^{1/2}\right\|_{L^p(\mathbf{R}^m)}\end{aligned}\tag{3.19}$$

holds for arbitrary functions $\{g_k(\cdot)\}_{k\in\mathbf{Z}}$ on $\mathbf{R}^m$.

Proof. We follow an argument employed in [10, 1]. Let us first consider the case $2\leq p<2\gamma/(2-\gamma)$. By duality there exists a nonnegative function b in $L^{(p/2)'}(\mathbf{R}^m)$ with $\|b\|_{(p/2)'}\leq 1$ such that

$$\left\|\Big(\sum_{k\in\mathbf{Z}}\int_{\theta^k}^{\theta^{k+1}}|\sigma_{t,\mathcal{P},h}*g_k|^2\,dt/t\Big)^{1/2}\right\|_{L^p(\mathbf{R}^m)}^2=\int_{\mathbf{R}^m}\sum_{k\in\mathbf{Z}}\int_{\theta^k}^{\theta^{k+1}}|\sigma_{t,\mathcal{P},h}*g_k(x)|^2\,b(x)\frac{dt}{t}dx.\tag{3.20}$$

By applying Schwarz's inequality we get

$$\begin{aligned}|\sigma_{t,\mathcal{P},h}*g_k(x)|^2 &\leq C\,\|h\|_{\Delta_\gamma}^\gamma\,\|\Omega\|_{L^q(\mathbf{S}^{n-1})}\\ &\qquad\times\Big(\int_{1/2t}^{t}\int_{\mathbf{S}^{n-1}}|g_k(x-\mathcal{P}(sy))|^2\,|\Omega(y)|\,|h(s)|^{2-\gamma}\,d\sigma(y)ds/s\Big).\end{aligned}$$

Therefore, by a change of variable we have

$$\begin{aligned}&\left\|\Big(\sum_{k\in\mathbf{Z}}\int_{\theta^k}^{\theta^{k+1}}|\sigma_{t,\mathcal{P},h}*g_k|^2\,dt/t\Big)^{1/2}\right\|_{L^p(\mathbf{R}^m)}^2\\ &\qquad\leq C\,\|h\|_{\Delta_\gamma}^\gamma\,\|\Omega\|_{L^q(\mathbf{S}^{n-1})}\int_{\mathbf{R}^m}\Big(\sum_{k\in\mathbf{Z}}|g_k(x)|^2\Big)M_{|h|^{2-\gamma},\theta}\tilde{b}(-x)dx,\end{aligned}\tag{3.21}$$

where $\tilde{b}(x)=b(-x)$. By a result of Stein and Wainger ([18, pp. 476–478]) we get that there exists a positive constant C_p which is independent of Ω, h and the coefficients of $P_1,...,P_m$ such that

$$\left\|\sigma^*_{\mathcal{P},h}(f)\right\|_p\leq C_p\,\|\Omega\|_{L^1(\mathbf{S}^{n-1})}\,\|h\|_{\Delta_\gamma}\,\|f\|_p\tag{3.22}$$

for $\gamma' < p \leq \infty$ and $f \in L^p(\mathbf{R}^m)$. By (3.21) and noticing that $|h(\cdot)|^{2-\gamma} \in \Delta_{\gamma/(2-\gamma)}(\mathbf{R}_+)$ and $(p/2)' > \left(\frac{\gamma}{2-\gamma}\right)'$ we obtain

$$\begin{aligned}\left\|M_{|h|^{2-\gamma},\theta,\mathcal{P}}(\tilde{b})\right\|_{L^{(p/2)'}(\mathbf{R}^m)} &\leq (q-1)^{-1}(\gamma-1)^{-1}\left\|\sigma^*_{\mathcal{P},h}(\tilde{b})\right\|_{L^{(p/2)'}(\mathbf{R}^m)} \\ &\leq C_p(q-1)^{-1}(\gamma-1)^{-1}\left\|h^{2-\gamma}\right\|_{\Delta_{\gamma/(2-\gamma)}} \\ &\quad \times \|\Omega\|_{L^q(\mathbf{S}^{n-1})}\|b\|_{L^{(p/2)'}(\mathbf{R}^m)} \\ &\leq C_p(q-1)^{-1}(\gamma-1)^{-1}\|h\|^{2-\gamma}_{\Delta_\gamma}\|\Omega\|_{L^q(\mathbf{S}^{n-1})}. \quad (3.23)\end{aligned}$$

Thus, by (3.21), (3.23) and Hölder's inequality we get (3.19) for $2 \leq p < 2\gamma/(2-\gamma)$. Let us now prove (3.19) for the case $2\gamma/(3\gamma-2) < p < 2$. By duality, there exist functions $f = f_k(x,t)$ defined on $\mathbf{R}^m \times \mathbf{R}_+$ with $\left\|\left\|\|f_k\|_{L^2([\theta^k,\theta^{k+1}],dt/t)}\right\|_{l^2}\right\|_{L^{p'}} \leq 1$ such that

$$\begin{aligned}&\left\|\Big(\sum_{k\in\mathbf{Z}}\int_{\theta^k}^{\theta^{k+1}}|\sigma_{t,\mathcal{P},h}*g_k|^2\,dt/t\Big)^{1/2}\right\|_p \\ &\quad= \int_{\mathbf{R}^m}\sum_{k\in\mathbf{Z}}\int_{\theta^k}^{\theta^{k+1}}(\sigma_{t,\mathcal{P},h}*g_k(x))\,f_k(x,t)\frac{dt}{t}dx \\ &\quad\leq C_p(q-1)^{-1/2}(\gamma-1)^{-1/2}\left\|\Big(\sum_{k\in\mathbf{Z}}|g_k|^2\Big)^{1/2}\right\|_p\|(H(f))^{1/2}\|_{p'}, \qquad (3.24)\end{aligned}$$

where

$$Hf(x) = \sum_{k\in\mathbf{Z}}\int_{\theta^k}^{\theta^{k+1}}|\sigma_{t,\mathcal{P},h}*f_k(x,t)|^2\,\frac{dt}{t}.$$

Now, since $p' > 2$, there exists a function $q \in L^{(p'/2)'}(\mathbf{R}^m)$ such that

$$\|H(f)\|_{p'/2} = \sum_{k\in\mathbf{Z}}\int_{\mathbf{R}^m}\int_{\theta^k}^{\theta^{k+1}}|f_k(x,t)*\sigma_{t,\mathcal{P},h}|^2\,\frac{dt}{t}q(x)dx.$$

By a similar argument as above, the choice of $f_k(x,t)$ and (3.21) we have

$$\begin{aligned}\|H(f)\|_{p'/2} &\leq C\|h\|^\gamma_{\Delta_\gamma}\|\Omega\|_{L^q(\mathbf{S}^{n-1})}\int_{\mathbf{R}^n}\sigma^*_{\mathcal{P},|h|^{2-\gamma}}(\tilde{q})(-x)\Big(\sum_{k\in\mathbf{Z}}\int_{\theta^k}^{\theta^{k+1}}|f_k(x,t)|^2\,dt/t\Big)dx \\ &\leq C\|h\|^\gamma_{\Delta_\gamma}\|\Omega\|_{L^q(\mathbf{S}^{n-1})}\left\|\Big(\sum_{k\in\mathbf{Z}}\int_{\theta^k}^{\theta^{k+1}}|f_k(\cdot,t)|^2\,dt/t\Big)\right\|_{p'/2}\|\sigma^*_{\mathcal{P},|h|^{2-\gamma}}(q)\|_{(p'/2)'} \\ &\leq C\|h\|^2_{\Delta_\gamma}\|\Omega\|^2_{L^q(\mathbf{S}^{n-1})}\end{aligned}$$

which in conjunction with (3.24) yields (3.19) for $2\gamma/(3\gamma-2) < p < 2$. The proof Lemma 3.7 is complete.

By following a similar argument as in the proof of Lemma 3.7 we get the following:

Lemma 3.8

Let $h \in \Delta_\gamma(\mathbf{R}_+)$ for some $\gamma \geq 2$, $\Omega \in L^q(\mathbf{S}^{n-1})$ for some $1 < q \leq 2$ and $\theta = 2^{q'\gamma'}$. Let $\mathcal{P} = (P_1, ..., P_m)$ be a polynomial mapping from $\mathbf{R}^n$ into $\mathbf{R}^m$. Then for any p satisfying $1 < p < \infty$ and $f \in L^p(\mathbf{R}^m)$, there exists a positive constant C_p which is independent of q, γ, Ω and h such that

$$\left\| \Big(\sum_{k\in\mathbf{Z}} \int_{\theta^k}^{\theta^{k+1}} |\sigma_{t,\mathcal{P},h} * g_k|^2 \, dt/t \Big)^{1/2} \right\|_{L^p(\mathbf{R}^m)}$$

$$\leq C_p (q-1)^{-1/2} \gamma^{1/2} \|\Omega\|_{L^q(\mathbf{S}^{n-1})} \|h\|_{\Delta_\gamma} \left\| \Big(\sum_{k\in\mathbf{Z}} |g_k|^2 \Big)^{1/2} \right\|_{L^p(\mathbf{R}^m)}$$

holds for arbitrary functions $\{g_k(\cdot)\}_{k\in\mathbf{Z}}$ on $\mathbf{R}^m$.

Lemma 3.9

Let $h \in \Delta_\gamma(\mathbf{R}_+)$ for some $1 < \gamma \leq 2$, $\Omega \in L^q(\mathbf{S}^{n-1})$ for some $1 < q \leq 2$ and $\theta = 2^{q'\gamma'}$. Suppose that $\mathcal{P} \in \mathcal{F}(n,m)$. Then for every $p, 1 < p < \infty$, there exists a positive constant C_p which is is independent of h, Ω, q and s such that

$$\|M_{h,\theta,\mathcal{P}}(f)\|_p \leq C_p (q-1)^{-1}(\gamma-1)^{-1} \|\Omega\|_{L^q(\mathbf{S}^{n-1})} \|h\|_{\Delta_\gamma} \|f\|_p; \tag{3.25}$$

$$\|\sigma^*_{\mathcal{P},h}(f)\|_p \leq C_p (q-1)^{-1}(\gamma-1)^{-1} \|\Omega\|_{L^q(\mathbf{S}^{n-1})} \|h\|_{\Delta_\gamma} \|f\|_p \tag{3.26}$$

for every $f \in L^p(\mathbf{R}^m)$. Furthermore, if $\mathcal{P}(-x) = -\mathcal{P}(x)$, then the constant C_p depends only on p, n, $m, \deg(\mathcal{P})$ and neither on the function Ω, h nor on the coefficients of the polynomial components of the mapping $\mathcal{P}$.

Proof. We shall start by proving (3.25). We may assume without loss of generality that $\Omega \geq 0$ and $h \geq 0$. We shall prove (3.25) by induction on $\deg(\mathcal{P})$. First, if $\deg(\mathcal{P}) = 0$, then

$$M_{h,\theta,\mathcal{P}} f(x) \leq C(q-1)^{-1}(\gamma-1)^{-1} \|\Omega\|_{L^1(\mathbf{S}^{n-1})} \|h\|_{\Delta_1} |f(x - \mathcal{P}(0))|$$

and hence (3.25) holds trivially. Next, assume that (3.25) holds for all $\mathcal{P} \in \mathcal{F}(n,m)$ with $\deg(\mathcal{P}) \leq d-1$.

Now suppose that $\deg(\mathcal{P}) = d$. Then $\mathcal{P} = H(x) + \mathcal{R}(x)$ for some non zero $\mathcal{H} \in \mathcal{F}_{n,m,d}$, $\mathcal{R} \in \mathcal{F}(n,m)$ and with $\deg(\mathcal{R}) \leq d-1$. By the inductive hypothesis we have

$$\|M_{h,\theta,\mathcal{R}}(f)\|_{L^p(\mathbf{R}^m)} \leq C_p (q-1)^{-1}(\gamma-1)^{-1} \|\Omega\|_{L^q(\mathbf{S}^{n-1})} \|h\|_{\Delta_\gamma} \|f\|_{L^p(\mathbf{R}^m)} \tag{3.27}$$

for $1 < p < \infty$ and $f \in L^p(\mathbf{R}^m)$.

Let $m_d = \dim(\mathcal{V}_d)$ and $L : \mathbf{R}^m \to \mathbf{R}^{m_d}$ be a linear transformation such that $\|(\xi \cdot \mathcal{H})(\cdot)\| = |L(\xi)|$ for $\xi \in \mathbf{R}^m$. Then by the proof of Lemma 3.6 and straightforward calculations we obtain

$$\|\sigma_{t,\mathcal{P},h}\| \leq C \|\Omega\|_{L^q(\mathbf{S}^{n-1})} \|h\|_{\Delta_\gamma}; \|\sigma_{t,\mathcal{R},h}\| \leq C \|\Omega\|_{L^q(\mathbf{S}^{n-1})} \|h\|_{\Delta_\gamma}; \tag{3.28}$$

$$|\hat{\sigma}_{t,\mathcal{P},h}(\xi)| \leq C \|\Omega\|_{L^q(\mathbf{S}^{n-1})} \|h\|_{\Delta_\gamma} \left(t^d |L(\xi)|\right)^{-1/(4d\gamma' q')}; \tag{3.29}$$

$$|\hat{\sigma}_{t,\mathcal{P},h}(\xi) - \hat{\sigma}_{t,\mathcal{R},h}(\xi)| \leq C \|\Omega\|_{L^q(\mathbf{S}^{n-1})} \|h\|_{\Delta_\gamma} \left(t^d |L(\xi)|\right)^{1/(4d\gamma' q')}. \tag{3.30}$$

By the arguments in the proof of Lemma 6.2 in [10], we may assume without loss of generality that $m_d \leq m$ and $L = \pi^m_{m_d}$ .Choose and fix a $\varphi \in \mathcal{S}(\mathbf{R}^{m_d})$ such that $\hat{\varphi}(x) = 1$ for $|x| \leq 1/2$ and $\hat{\varphi}(x) = 0$ for $|x| \geq 1$. For each $t \in \mathbf{R}_+$, let $(\varphi_t)\hat{}(x) = \hat{\varphi}(tx)$. Define the family of measures $\{\Upsilon_t\}_{t\in\mathbf{R}_+}$ and the sequence of measures$\{\hat{\vartheta}_k\}_{k\in\mathbf{Z}}$ by

$$\hat{\Upsilon}_t(\xi) = \hat{\sigma}_{t,\mathcal{P},h}(\xi) - \hat{\sigma}_{t,\mathcal{R},h}(\xi)\,(\varphi_t)\hat{}(\pi^m_{m_d}\xi) \text{ and } \hat{\lambda}_k(\xi) = \int_{\theta^k}^{\theta^{k+1}} \hat{\Upsilon}_t(\xi)dt/t. \tag{3.31}$$

By (3.28)–(3.31) and the choice of φ we obtain

$$\left|\hat{\lambda}_k(\xi)\right| \leq C(q-1)^{-1}(\gamma-1)^{-1}\|\Omega\|_{L^q(\mathbf{S}^{n-1})}\|h\|_{\Delta_\gamma}\left|2^{(kdq'\gamma')}\pi^m_{m_d}\xi\right|^{\pm 1/(4dq'\gamma')}. \tag{3.32}$$

Let

$$g(f) = \Big(\sum_{k\in\mathbf{Z}}|\lambda_k * f|^2\Big)^{1/2} \text{ and } \lambda^*(f) = \sup_{k\in\mathbf{Z}}||\lambda_k| * f|\,.$$

Then by (3.31) we have

$$M_{h,\theta,\mathcal{P}}f(x) \leq g(f)(x) + C(\mathcal{M}_{m_d}\otimes id_{\mathbf{R}^{m-m_d}})\,(M_{h,\theta,\mathcal{R}}f(x))\,, \tag{3.33}$$

$$\lambda^* f(x) \leq g(f)(x) + 2C[(\mathcal{M}_{m_d}\otimes id_{\mathbf{R}^{m-m_d}})]\,(M_{h,\theta,\mathcal{R}}f(x))\,, \tag{3.34}$$

where $\mathcal{M}_s$ denotes the Hardy-Littlewood maximal function on $\mathbf{R}^s$. It follows from (3.32) and Plancherel's theorem that

$$\|g(f)\|_{L^2} \leq C(q-1)^{-1}(\gamma-1)^{-1}\|\Omega\|_{L^q(\mathbf{S}^{n-1})}\|h\|_{\Delta_\gamma}\|f\|_{L^2}\,. \tag{3.35}$$

By the L^p boundedness of the Hardy-Littlewood maximal function, (3.27) and (3.34), (3.35) we get

$$\|\lambda^*(f)\|_2 \leq C(q-1)^{-1}(\gamma-1)^{-1}\|\Omega\|_{L^q(\mathbf{S}^{n-1})}\|h\|_{\Delta_\gamma}\|f\|_2 \tag{3.36}$$

for some positive constant C independent of Ω, h, q and γ. By using (the proof of) the lemma in [9, p. 544] with $p_0 = 4$ and $q = 2$ and the estimate $\|\lambda_k\| \leq C(q-1)^{-1}(\gamma-1)^{-1}\|\Omega\|_{L^q(\mathbf{S}^{n-1})}\|h\|_{\Delta_\gamma}$ we obtain

$$\Big\|\Big(\sum_{k\in\mathbf{Z}}|\lambda_k * g_k|^2\Big)^{1/2}\Big\|_{p_0} \leq C_{p_0}(q-1)^{-1}(\gamma-1)^{-1}\|\Omega\|_{L^q(\mathbf{S}^{n-1})}\|h\|_{\Delta_\gamma}\Big\|\Big(\sum_{k\in\mathbf{Z}}|g_k|^2\Big)^{1/2}\Big\|_{p_0} \tag{3.37}$$

if $1/4 = |1/p_0 - 1/2|$. Now, by (3.32), (3.37) and invoking Lemma 3.3 we get

$$\|g(f)\|_{L^p} \leq C_p(q-1)^{-1}(\gamma-1)^{-1}\|\Omega\|_{L^q(\mathbf{S}^{n-1})}\|h\|_{\Delta_\gamma}\|f\|_{L^p} \tag{3.38}$$

for $p \in (\frac{4}{3}, 4)$. By the L^p boundedness of the Hardy-Littlewood maximal function, (3.27), (3.34) and (3.38) we get

$$\|\lambda^*(f)\|_{L^p} \leq C(q-1)^{-1}(\gamma-1)^{-1}\|\Omega\|_{L^q(\mathbf{S}^{n-1})}\|h\|_{\Delta_\gamma}\|f\|_{L^p} \tag{3.39}$$

for $p \in (\frac{4}{3}, 4)$. Reasoning as above, (3.32), (3.37), (3.39) and Lemma 3.3 provide

$$\|g(f)\|_{L^p} \le C_p (q-1)^{-1} (\gamma-1)^{-1} \|\Omega\|_{L^q(\mathbf{S}^{n-1})} \|h\|_{\Delta_\gamma} \|f\|_{L^p} \tag{3.40}$$

for $p \in (\frac{8}{7}, 8)$. By using this argument repeatedly we ultimately obtain that

$$\|g(f)\|_{L^p} \le C_p (q-1)^{-1} (\gamma-1)^{-1} \|\Omega\|_{L^q(\mathbf{S}^{n-1})} \|h\|_{\Delta_\gamma} \|f\|_{L^p} \tag{3.41}$$

for $p \in (1, \infty)$. Therefore, by the L^p boundedness of the Hardy-Littlewood maximal function, (3.27), (3.33) and (3.41) we conclude that (3.2) holds for $p \in (1, \infty)$. Also, (3.2) holds trivially for $p = \infty$. Finally, if $\mathcal{P}(-x) = -\mathcal{P}(x)$, then at each step of our inductive argument d is always an odd number. Therefore, by Lemma 3.2 and the above argument, the constant C_p depends only p, n, m, $\deg(\mathcal{P})$ and neither on the functions Ω and h nor on the coefficients of the polynomial components of the mapping $\mathcal{P}$. This concludes the proof of (3.25). Now we turn to the proof of (3.26). It is easy to see that the following inequality holds:

$$\sigma^*_{\mathcal{P},h} f(x) \le 2 \sup_{k\in\mathbb{Z}} \Big(\int_{\theta^k < |u| \le \theta^{k+1}} |f(x - \mathcal{P}(u))| \frac{|\Omega(u')|}{|u|^n} du \Big). \tag{3.42}$$

Thus the proof of (3.26) follows by the last inequality and adapting a similar argument as in the proof of (3.25).

By Lemma 3.9 and following argument similar to the one used in the proof of Lemma 3.7 we get the following:

Lemma 3.10

Let $h \in \Delta_\gamma(\mathbf{R}_+)$ for some $1 < \gamma \le 2$, $\Omega \in L^q(\mathbf{S}^{n-1})$ for some $1 < q \le 2$ and $\theta = 2^{q'\gamma'}$. Suppose that $\mathcal{P} \in \mathcal{F}(n, m)$. Then for any $p \in (1, \infty)$ and $f \in L^p(\mathbf{R}^m)$, there exists a positive constant C_p which is independent of q, γ, Ω and h such that the following inequalities

(a) $$\Big\| \Big(\sum_{k\in\mathbf{Z}} \int_{\theta^k}^{\theta^{k+1}} |\sigma_{t,\mathcal{P},h} * g_k|^2 dt/t \Big)^{1/2} \Big\|_{L^p(\mathbf{R}^m)} \le C_p (q-1)^{-1/2} (\gamma-1)^{-1/2} \times \|\Omega\|_{L^q(\mathbf{S}^{n-1})} \|h\|_{\Delta_\gamma} \Big\| \Big(\sum_{k\in\mathbf{Z}} |g_k|^2 \Big)^{1/2} \Big\|_{L^p(\mathbf{R}^m)} \text{ for } 2 \le p < \infty;$$

(b) $$\Big\| \Big(\sum_{k\in\mathbf{Z}} \int_{\theta^k}^{\theta^{k+1}} |\sigma_{t,\mathcal{P},h} * g_k|^2 dt/t \Big)^{1/2} \Big\|_{L^p(\mathbf{R}^m)} \le C_p (q-1)^{-1} (\gamma-1)^{-1} \|\Omega\|_{L^q(\mathbf{S}^{n-1})} \|h\|_{\Delta_\gamma} \Big\| \Big(\sum_{k\in\mathbf{Z}} |g_k|^2 \Big)^{1/2} \Big\|_{L^p(\mathbf{R}^m)} \text{ for } 1 < p < 2$$

hold for arbitrary functions $\{g_k(\cdot)\}_{k\in\mathbf{Z}}$ on $\mathbf{R}^m$.

4. Proof of the main results

Proof of Theorem 2.1 Let first assume that $h \in \Delta_\gamma(\mathbf{R}_+)$ for some $\gamma \in (1,2]$. By Minkowski's inequality we have

$$\begin{aligned}\mathcal{M}^\rho_{\mathcal{P},\Omega,h}f(x) &= \Big(\int_0^\infty \Big|\sum_{k=0}^\infty \frac{1}{t^\rho}\int_{2^{-k-1}t<|y|\le 2^{-k}t} f(x-\mathcal{P}(u))\frac{\Omega(u')}{|u|^{n-\rho}}h(|u|)du\Big|^2 dt/t\Big)^{1/2}\\ &\le \sum_{k=0}^\infty \Big(\int_0^\infty \Big|\frac{1}{t^\rho}\int_{2^{-k-1}t<|y|\le 2^{-k}t} f(x-\mathcal{P}(u))\frac{\Omega(u')}{|u|^{n-\rho}}h(|u|)du\Big|^2 dt/t\Big)^{1/2}\\ &= (1-2^{-\sigma})^{-1}\mathcal{S}^\rho_{\mathcal{P},\Omega,h}f(x), \qquad (4.1)\end{aligned}$$

where

$$\mathcal{S}^\rho_{\mathcal{P},\Omega,h}f(x) = \Big(\int_0^\infty \Big|\frac{1}{t^\rho}\int_{1/2t<|y|\le t} f(x-\mathcal{P}(u))\frac{\Omega(u')}{|u|^{n-\rho}}h(|u|)du\Big|^2 dt/t\Big)^{1/2}.$$

Let $0<n_1<n_2<\cdots<n_{\tilde N}=\deg(\mathcal{P})$ be non-negative integers, and polynomials $\{P^l_\nu : 1\le \nu\le N, 1\le l\le \tilde N\}$ such that for $x\in\mathbf{R}^n, \mathcal{P}(x)=\sum_{l=1}^{\tilde N}\mathcal{P}^l(x)+A(|x|)$, where $\mathcal{P}^l(x)=(P^l_1(x),\dots,P^l_N(x))\in(\mathcal{H}_{n,n_l})^N$, $A(t)=(A_1(t),...,A_N(t))$ with $t\in\mathbf{R}$, $Z^n_{n_l}(P^l_\nu)=P^l_\nu$, and $A_\nu\in\mathcal{A}_1$ for $1\le\nu\le N$ and $1\le l\le\tilde N$. For $1\le l\le\tilde N$, let δ_l denote the number of elements of $\{\beta\in(\mathbf{N}\cup\{0\})^n$: $|\beta|=n_l\}$ and write $\{\beta\in(\mathbf{N}\cup\{0\})^n : |\beta|=n_l\}=\{\beta(1),\dots,\beta(\delta_l)\}$. Write $P^l_j(x)=\sum_{k=1}^{\delta_l}\eta_{k,j}x^{\beta(k)}$ and define the linear mappings $L_l:\mathbf{R}^N\to\mathbf{R}^{\delta_l}$ by $L_l(\xi)=(\sum_{j=1}^m\eta^l_{1,j}\xi_j,\dots,\sum_{j=1}^m\eta^l_{\delta_l,j}\xi_j)$ for $1\le j\le N, 1\le l\le\tilde N$. Let $\Phi_l(x)=\sum_{j=1}^l\mathcal{P}^j(x)+\mathcal{W}(|x|)$ for $1\le l\le\tilde N$ and $\Phi_0(x)=\mathcal{W}(|x|)$. For simplicity, let $\sigma^{(l)}_{t,h}=\sigma_{t,\Phi_l,h}$ and $\sigma^{(l)*}_h f(x)=\sigma^*_{\Phi_l,h}f(x)$ for $1\le l\le\tilde N$. Let $\theta=2^{q'\gamma'}$. By definition of $\sigma^{(l)}_{t,h}$ and invoking Lemma 3.6, it is easy to verify that

$$\Big(\int_{\theta^k}^{\theta^{k+1}}\Big|\hat\sigma^{(l)}_{t,h}(\xi)\Big|^2 dt/t\Big)^{1/2} \le C(q-1)^{-1/2}(\gamma-1)^{-1/2}\|\Omega\|_{L^q(\mathbf{S}^{n-1})}\|h\|_{\Delta_\gamma} \text{ for } 1\le l\le\tilde N; \qquad (4.2)$$

$$\Big(\int_{\theta^k}^{\theta^{k+1}}\Big|\hat\sigma^{(l)}_{t,h}(\xi)\Big|^2 dt/t\Big)^{1/2} \le C(q-1)^{-1/2}(\gamma-1)^{-1/2}\|\Omega\|_{L^q(\mathbf{S}^{n-1})}\|h\|_{\Delta_\gamma}\Big|\theta^{n_lk}L_l(\xi)\Big|^{-1/(4n_l\gamma'q')}. \qquad (4.3)$$

Also, by a change of variable we have

$$\Big(\int_{\theta^k}^{\theta^{k+1}}\Big|\hat\sigma^{(l)}_{t,h}(\xi)-\hat\sigma^{(l-1)}_{t,h}(\xi)\Big|^2 dt/t\Big)^{1/2} \le C(q-1)^{-1/2}(\gamma-1)^{-1/2}\|\Omega\|_{L^q(\mathbf{S}^{n-1})}\|h\|_{\Delta_\gamma}\Big|\theta^{n_lk}L_l(\xi)\Big|.$$

By the trivial estimate (4.2) and the last estimate we obtain

$$\Big(\int_{\theta^k}^{\theta^{k+1}}\Big|\hat\sigma^{(l)}_{t,h}(\xi)-\hat\sigma^{(l-1)}_{t,h}(\xi)\Big|^2 dt/t\Big)^{1/2} \le C(q-1)^{-1/2}(\gamma-1)^{-1/2}\|\Omega\|_{L^q(\mathbf{S}^{n-1})}\|h\|_{\Delta_\gamma}\Big|\theta^{n_lk}L_l(\xi)\Big|^{1/(4n_l\gamma'q')}. \qquad (4.4)$$

Also, by Lemma 3.7 we get for any p satisfying $|1/p - 1/2| < 1/\gamma'$, there exists a positive constant C_p which is independent of θ, Ω and h such

$$\left\| \Big(\sum_{k \in \mathbf{Z}} \int_{\theta^k}^{\theta^{k+1}} \left| \sigma_{t,h}^{(l)} * g_k \right|^2 dt/t \Big)^{1/2} \right\|_{L^p(\mathbf{R}^m)} \leq C_p (q-1)^{-1/2} (\gamma - 1)^{-1/2} \|\Omega\|_{L^q(\mathbf{S}^{n-1})} \|h\|_{\Delta_\gamma} \left\| \Big(\sum_{k \in \mathbf{Z}} |g_k|^2 \Big)^{1/2} \right\|_{L^p(\mathbf{R}^m)} \quad (4.5)$$

for arbitrary functions $\{g_k(\cdot)\}_{k \in \mathbf{Z}}$ on $\mathbf{R}^m$. By (4.1) we have

$$\begin{aligned} \mathcal{M}_{\mathcal{P},\Omega,h}^{\rho} f(x) &\leq \left(1 - 2^{-\sigma}\right)^{-1} \mathcal{S}_{\mathcal{P},\Omega,h}^{\rho} f(x) \\ &= \left(1 - 2^{-\sigma}\right)^{-1} \Big(\int_0^\infty \left| \sigma_{t,h}^{(\tilde{N})} * f(x) \right|^2 dt/t \Big)^{1/2}. \end{aligned} \quad (4.6)$$

By (4.2)–(4.6) and applying Lemma 3.5 we get (2.1) for the case $h \in \Delta_\gamma(\mathbf{R}_+)$ for some $1 < \gamma \leq 2$. Finally the proof of (2.1) for the case $h \in \Delta_\gamma(\mathbf{R}_+)$ for some $\gamma \geq 2$ follows by the same argument as above except that we need to invoke Lemma 3.8 instead of Lemma 3.7. Thus the proof of Theorem 2.1 is complete.

Proof of Theorem 2.2 Since $\mathcal{P} \in \mathcal{F}(n,m)$, there are integers $0 < n_1 < n_2 < \cdots < n_N = \deg(\mathcal{P})$, and nonzero $\mathcal{P}^l \in \mathcal{F}_{n,m,n_d}$ for $1 \leq l \leq N$ such that $\mathcal{P}(x) = \mathcal{P}(0) + \sum_{l=1}^N \mathcal{P}^l(x)$. Let $\Phi_0(x) = \mathcal{P}(0)$ and $\Phi_l(x) = \mathcal{P}(0) + \sum_{j=1}^l \mathcal{P}^j(x)$ for $1 \leq l \leq N$. Now, the rest of the proof of Theorem 2.2 follows the same argument as in the proof of Theorem 2.1 and invoking Lemma 3.10 instead of Lemma 3.9. Details will be omitted.

Proof of Theorem 2.3 (a) We follow the extrapolation method of Yano (see [21, 23, Chapter XII, p. 119–120]). Assume $\Omega \in L(\log L)^{1/2}(\mathbf{S}^{n-1})$ and satisfies (1.1). Let us now fix $\mathcal{P}, h \in \Delta_\gamma(\mathbf{R}_+)$ for $\gamma > 1$ with $\|h\|_{\Delta_\gamma} \leq 1, p$ satisfies $|1/p - 1/2| < \min\{1/2, 1/\gamma'\}$ and a function f with $\|f\|_p \leq 1$. Let $R(\Omega) = \left\|\mathcal{M}_{\mathcal{P},\Omega,h}^{\rho}(f)\right\|_p$. Then we have $R(\Omega_1 + \Omega_2) \leq R(\Omega_1) + R(\Omega_2)$. Now, we decompose Ω as follows: For $m \in \mathbf{N}$, let $\mathbf{J}_m = \{x \in \mathbf{S}^{n-1} : 2^m \leq |\Omega(x)| < 2^{m+1}\}$. For $m \in \mathbf{N}$, set $b_m = \Omega \chi_{\mathbf{J}_m}$, where $\chi_{\mathbf{A}}$ is the characteristic function of a set A. Set $I(\Omega) = \{m \in \mathbf{N} : \|b_m\|_1 \geq 2^{-4m}\}$ and define the sequence of functions $\{\Omega_m\}_{m \in I(\Omega) \cup \{0\}}$ by

$$\begin{aligned} \Omega_m(x) &= \|b_m\|_1^{-1} \Big(b_m(x) - \int_{\mathbf{S}^{n-1}} b_m(x) d\sigma(x) \Big) \text{ for } m \in I(\Omega), \\ \Omega_0(x) &= \Omega(x) - \sum_{m \in I(\Omega)} \|b_m\|_1 \ \Omega_m(x). \end{aligned}$$

Now, it is easy to see that

$$\int_{\mathbf{S}^{n-1}} \Omega_m(u) \, d\sigma(u) = 0. \quad (4.7)$$

Next, we notice that if $m \in \mathbf{N}$, $x \in \mathbf{J}_m$, we have $m \leq \frac{1}{\log 2} \log(2 + |\Omega(x)|)$ which in turn easily implies

$$\sum_{m \in I_\Omega} m^{1/2} \|b_m\|_1 \leq \frac{1}{\sqrt{\log 2}} \|\Omega\|_{L(\log L)^{1/2}(\mathbf{S}^{n-1})}. \quad (4.8)$$

Also, by the definition of I_Ω and straightforward computations we have

$$\|\Omega_m\|_{1+1/m} \le 2^6 \text{ for } m \in I_\Omega \text{ and } \|\Omega_0\|_2 \le 2^2. \tag{4.9}$$

Therefore, by (2.1), (4.7)–(4.9) we have

$$\begin{aligned} R(\Omega) &\le R(\Omega_0) + \sum_{m\in I(\Omega)} \|b_m\|_1 R(\Omega_m) \\ &\le C_p A(\gamma)\Big(\|\Omega_0\|_{L^2(\mathbf{S}^{n-1})} + \sum_{m\in I(\Omega)} m^{1/2} \|b_m\|_1 \|\Omega_m\|_{1+1/m} \Big) \\ &\le C_p A(\gamma) \left(1 + \|\Omega\|_{L(\log L)^{1/2}(\mathbf{S}^{n-1})}\right). \end{aligned}$$

Proof of Theorem 2.3 (b) Assume that $\Omega \in B_q^{(0,-1/2)}(\mathbf{S}^{n-1})$ for some $q > 1$ and satisfies (1.1). We may assume without loss of generality we may assume $1 < q \le 2$. Fix $\mathcal{P}, h \in \Delta_\gamma(\mathbf{R}_+)$ for $\gamma > 1$ with $\|h\|_{\Delta_\gamma} \le 1, p$ satisfies $|1/p - 1/2| < \min\{1/2, 1/\gamma'\}$ and a function f with $\|f\|_p \le 1$. Let $X(\Omega) = \|\mathcal{M}^\rho_{\mathcal{P},\Omega,h}(f)\|_p$. Since $\Omega \in B_q^{(0,-1/2)}(\mathbf{S}^{n-1})$, we can write Ω as $\Omega = \sum_{\mu=1}^\infty \lambda_\mu b_\mu$, where $\lambda_\mu \in \mathbf{C}$, b_μ is a q-block supported on a cap I_μ on $\mathbf{S}^{n-1}$ and $M_q^{(0,-1/2)}(\{\lambda_\mu\}) < \infty$. To each block function $b_\mu(\cdot)$, let $\tilde{\Omega}_\mu(\cdot)$ be a function defined by

$$\tilde{\Omega}_\mu(x) = b_\mu(x) - \int_{\mathbf{S}^{n-1}} b_\mu(u) d\sigma(u).$$

Let $\mathbf{K} = \{\mu \in \mathbf{N} : |I_\mu| < e^{-(q-1)^{-1}}\}$ and let $\tilde{\Omega}_0 = \Omega - \sum_{\mu\in\mathbf{K}}^\infty \lambda_\mu \tilde{\Omega}_\mu$. Also, for $\mu \in \mathbf{K}$ we let $\alpha_\mu = \log(|I_\mu|^{-1})$ and $\beta = \sum_{\mu=1}^\infty |\lambda_\mu|$. Then it is easy to see that

$$\Omega = \tilde{\Omega}_0 + \sum_{\mu\in\mathbf{K}}^\infty \lambda_\mu \tilde{\Omega}_\mu; \tag{4.10}$$

$$\int_{\mathbf{S}^{n-1}} \tilde{\Omega}_\mu(u)\, d\sigma(u) = 0 \text{ for all } \mu \in \mathbf{K} \cup \{0\}; \tag{4.11}$$

$$\left\|\tilde{\Omega}_0\right\|_q \le \beta e^{1/q}. \tag{4.12}$$

Also, for $\mu \in \mathbf{K}$ we have $1 + \frac{1}{\alpha_\mu} < q$ and hence by Hölder's inequality we have

$$\left\|\tilde{\Omega}_\mu\right\|_{1+\frac{1}{\alpha_\mu}} \le 2\|b_\mu\|_q |I_\mu|^{\frac{q-1-\frac{1}{\alpha_\mu}}{q\left(1+\frac{1}{\alpha_\mu}\right)}} \le 2\left(|I_\mu|^{-\frac{1}{q'}}\right)|I_\mu|^{\frac{q-1-\frac{1}{\alpha_\mu}}{q\left(1+\frac{1}{\alpha_\mu}\right)}} = 2|I_\mu|^{-\frac{1}{\alpha_\mu+1}} \le 4. \tag{4.13}$$

By (4.10)–(4.13) and invoking Theorem 2.1 we get

$$\begin{aligned} X(\Omega) &\le X(\tilde{\Omega}_0) + \sum_{\mu\in\mathbf{K}} |\lambda_\mu| X(\tilde{\Omega}_\mu) \\ &\le C_p A(\gamma)\Big((q-1)^{-1/2}\|\tilde{\Omega}_0\|_q + \sum_{\mu\in\mathbf{K}} |\lambda_\mu| \left(\log|I_\mu|^{-1}\right)^{1/2} \|\tilde{\Omega}_\mu\|_{1+1/\alpha_\mu}\Big) \\ &\le C_p A(\gamma)\Big(\beta e^{1/q}(q-1)^{-1/2} + 4\sum_{\mu\in\mathbf{K}} \left(|\lambda_\mu| \left(\log|I_\mu|^{-1}\right)^{1/2}\right)\Big) \\ &\le C_p A(\gamma)(1 + \|\Omega\|_{B_q^{(0,-1/2)}(\mathbf{S}^{n-1})}). \end{aligned}$$

Proof of Theorem 2.4 We start with a proof of Theorem 2.4 (a). Fix $\mathcal{P} \in \mathcal{F}(n,m)$, $q \in (1,2]$, $\Omega \in L^q(\mathbf{S}^{n-1})$, $p \in [2,\infty)$ and a function f with $\|f\|_p \leq 1$. Let $B(h) = \|\mathcal{M}^{\rho}_{\mathcal{P},\Omega,h}(f)\|_p$. Decompose h as follows: For $m \in \mathbf{N}$, let $\mathbf{E}_m = \{x \in \mathbf{R}_+ : 2^m \leq |h(x)| < 2^{m+1}\}$. For $m \in \mathbf{N}$, set $h_m = h\chi_{\mathbf{E}_m}$ and set $D(h) = \{m \in \mathbf{N} : d_m(h) \geq 2^{-4m}\}$. Also, let $h_0 = h - \sum_{m\in D(h)} h_m$. Then it is easy to verify that

$$\|h_m\|_{\Delta_{1+1/m}} \leq 2^m (d_m(h))^{m/(m+1)} \leq 2^m d_m(h); \tag{4.14}$$

$$\|h_0\|_{\Delta_2} \leq 32. \tag{4.15}$$

By (4.14)–(4.15) and invoking Theorem 2.2 we get

$$\begin{aligned} B(h) &\leq B(h_0) + \sum_{m\in D(h)} d_m(h) B(h_m) \\ &\leq C_p (q-1)^{-1/2}\Big(32 + \sum_{m\in D(h)} m^{1/2} 2^m d_m(h)\Big) \\ &\leq C_p (q-1)^{-1/2} \|\Omega\|_q (1 + N_{1/2}(h)). \end{aligned}$$

Next, fix $h \in \mathcal{N}_{1/2}, p \in [2,\infty)$ and a function f with $\|f\|_p \leq 1$. Let $S(\Omega) = \|\mathcal{M}^{\rho}_{\mathcal{P},\Omega,h}(f)\|_p$. Thus we have

$$S(\Omega) \leq C_p (1 + N_{1/2}(h))(q-1)^{-1/2} \|\Omega\|_q .$$

Now we follow the same argument as in the proof of Theorem 2.3 (a). Details will be omitted.

A proof of Theorem 2.3 (b) can be constructed by the following a similar argument as in the proof of Theorem 2.4 (a). Again we omit the details.

Proof of Theorem 2.5 By the same arguments as in the proof of Theorems 2.4 and 2.3 (a). Details will be omitted.

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